On Iterative Learning Control of Parabolic Distributed Parameter Systems

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***Abstract*— The Iterative Learning Control (ILC) technique is extended to distributed parameter systems governed by parabolic partial differential equations (PDEs). ILC arises as an effective method to approach constrained optimization problems in PDE systems. We discuss both P-type and D-type ILC schemes for a distributed parameter system formulated as a general linear system $\Sigma(A, B, C, D)$ on a Hilbert space, in which the system operator A generates a strongly continuous semigroup. Under the assumption of identical initialization condition (IIC), conditions on the learning parameters are obtained to guarantee convergence of the P-type and D-type ILC schemes. Numerical simulations are presented for a 1D heat conduction control problem solved using ILC based on semigroup analysis. The numerical results show the effectiveness of the proposed ILC schemes.**

I. Introduction

Learning is one of the most defining characteristics of human beings. Based on the acquisition of knowledge in practice, new capacities and skills can be developed. Repetition and correction are commonly used by human beings as learning mechanisms. It is not a surprise then to find these mechanisms employed in control system design. When the system dynamics is not well known, trial experiments are usually implemented to learn about the system behavior or response. Observations arising from the experimental results can be compared with the desired behavior of the system to adjust external inputs or tune parameters in subsequent experiments until a satisfactory response is achieved. Iterative Learning Control (ILC) is a relatively new control technique originally proposed for the operation of mechanical robots carrying out repetitive trajectory tracking tasks (see, e.g., [1]). Due to the repetitiveness of the operations, the control and tracking error signals can be recorded during each repetition cycle and used to update the input signals to be applied during the following cycle. A successful ILC scheme can improve the tracking accuracy by adjusting the system inputs from one repetition cycle to another based on the error observations in each cycle. The theoretical framework of ILC is based on the contraction mapping and fixed point theorems [2], which guarantee the convergence of ILC schemes.

ILC has been widely investigated for finite dimensional systems (see, e.g., [3], [4], [5], [6], [7] and references therein). However, there are very limited studies on ILC for distributed parameter systems governed by partial differential equations (PDEs). In [8], ILC is applied to a temporal-spatial discretized first order hyperbolic partial PDE, guaranteeing stability of the closed loop system and satisfying performance requirements. In [9], ILC is applied to a distributed parameter system which is governed by a second order hyperbolic PDE. In this work, we extend the ILC framework to distributed parameter systems governed by parabolic PDEs. Many infinite-dimensional dynamical systems can be described by parabolic PDEs, including processes such as heat and mass transfer, convection, diffusion, and transport (see, e.g., [10], [11] for more examples).

In the control of parabolic PDE systems, repetition and correction mechanisms are as common as in lumped parameter systems modeled by ordinary differential equations (ODEs). ILC arises as an effective method to approach constrained optimization problems in PDE systems with a repetitive behavior. One example is the control of density, temperature, current and momentum spatial profiles in tokamak plasmas, whose dynamics are governed by a set of coupled nonlinear parabolic PDEs (see, e.g., [12], [13], [14]). The use of transformer action to produce the toroidal plasma current means that present tokamaks operate in a pulsed mode. Each one of these pulses is called a discharge. One approach to plasma profile control focuses on creating the desired profiles during the plasma current ramp-up and early flattop phases of the discharge with the aim of maintaining these target profiles during the subsequent phases. Control actuators such as magnetic fields (magnetic control) and particle/wave injection (kinetic control) can be used to achieve the desired profiles. Since these actuators are constrained, experiments have shown that some of the desirable target profiles may not be achieved for all arbitrary initial conditions. In practice, the objective is to achieve the best possible approximate matching within a short time window during the early flattop phase of the total plasma current pulse. Thus, such matching problem can be formulated as a finite-time, optimal, PDE control problem. ILC could potentially be used to correct the control actuation discharge after discharge in order to minimize the matching error.

Another example arises in inverse source seeking problems where the central task is to find the system input that generates an evolutionary trajectory which in turns matches an observed output. Applications include heat transfer [15] and option pricing of financial derivatives [16]. There are two main approaches for inverse problems: numerical optimization [17] and statistical inverse computation [18]. The first approach formulates constrained optimization problems in which the difference between observed and numerically predicted evolutions must be minimized by properly choos-

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ing the control input. Optimization constraints are usually imposed by absolute and rate bounds for the control input and by the dynamics of the evolutionary PDE system. Nonlinear programming algorithms such as the sequential quadratic programming (SQP) method are often used to generate iterative computations. In each computational iteration, the optimization result from the previous iteration is updated by the nonlinear programming searching result. The computational iterations continue until a predefined error tolerance criterion is satisfied. This is a typical repetitive process that could be solved by an ILC scheme.

The organization of this paper is as follows. In Section II, we extend the ILC scheme to a distributed parameter system formulated as a general abstract linear system on a Hilbert space. The effectiveness of the proposed method is illustrated in Section III through numerical simulations. We finish the paper in Section IV by stating conclusions and discussing future research topics.

II. ILC on Infinite Dimensional Spaces

We consider a linear distributed parameter system which can be formulated as the following general linear system $\Sigma(A,B,C,D)$ on a Hilbert space $\mathcal{X}$ [19]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, x(0) = x_0, \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

where $x \in \mathcal{X}$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ ($\mathcal{U}$ and $\mathcal{Y}$ are also Hilbert spaces). The system operator A is assumed to be the infinitesimal generator of a strongly continuous semigroup (also denoted by C_0-semigroup) $\mathbb{T}(t)$ on the Hilbert space $\mathcal{X}$. The C_0-semigroup $\mathbb{T}(t)$ plays in infinite dimensional systems the same role that the transition matrix $\exp(At)$ plays in finite dimensional systems. The input operator B is a bounded linear operator from $\mathcal{U}$ to $\mathcal{X}$, which is denoted by $B \in \mathcal{L}(\mathcal{U},\mathcal{X})$. Similarly, we assume $C \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U},\mathcal{Y})$. Given an initial value $x_0 \in \mathcal{X}$ and an admissible control function $u \in \mathcal{U}$, the state of (1) can be represented by the following formula (also called "mild solution")

$$x(t) = \mathbb{T}(t)x_0 + \int_0^t \mathbb{T}(t-\tau)Bu(\tau)d\tau, \tag{2}$$

$$y(t) = C\mathbb{T}(t)x_0 + \int_0^t C\mathbb{T}(t-\tau)Bu(\tau)d\tau + Du(t). \tag{3}$$

We first discuss the representation of a C_0-semigroup. In the scenario of finite dimensional systems, for a general matrix A, we can compute its eigenvalue problem to obtain the pairs (λ_n, ϕ_n) satisfying $A\phi_n = \lambda_n\phi_n$. For any eigenvalue λ_n with non-unique algebraic multiplicity ($r_n > 1$), i.e., $(A - \lambda_n I)^{r_n}\phi_n = 0$, we can take advantage of the generalized eigenvectors to form a complete eigenvector basis. Namely, we obtain the r_n generalized eigenvectors associated with λ_n by solving a sequence of linear equations $(A - \lambda_n I)\tilde{\phi}_{n,j} = \tilde{\phi}_{n,j-1}$, where $j = 1, 2, \ldots, r_n$ and $\phi_{n,0} = 0$, and renumbering the eigenvectors $\tilde{\phi}_{n,1}, \ldots, \tilde{\phi}_{n,r_n}$ as $\phi_n, \ldots, \phi_{n+r_n-1}$. We can then expand a general vector x in terms of the eigenvectors $\{\phi_n\}_{n=1}^N$ with $N = \mathrm{Dim}(A)$, e.g., $x = \sum_{n=1}^N \langle x, \phi_n\rangle\phi_n$. Thus, we can obtain a representation of the matrix operation Ax, $Ax = A\sum_{n=1}^N \langle x, \phi_n\rangle\phi_n = \sum_{n=1}^N \lambda_n\langle x, \phi_n\rangle\phi_n$ and also a representation of the transition matrix, $\exp(At)x = \sum_{n=1}^N \exp(\lambda_n t)\langle x, \phi_n\rangle\phi_n$. Analogously, for a general infinitesimal generator of a C_0-semigroup, we have where the representations of the semigroup generator A and the semigroup $\mathbb{T}(t)$ are given by

$$Ax = \sum_{n=1}^{\infty} \lambda_n\langle x, \phi_n\rangle\phi_n, \ \mathbb{T}(t)x = \sum_{n=1}^{\infty} \exp(\lambda_n t)\langle x, \phi_n\rangle\phi_n.$$

The bound estimate is one of the most important results of C_0-semigroups: $\forall\omega > \omega_0 = \sup_n \lambda_n$, there exists a positive constant M_ω such that $\forall t > 0$ and the following estimate holds:

$$\|\mathbb{T}(t)\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \leq M_\omega \exp(\omega t), \tag{4}$$

where $\omega_0 = \sup_n \lambda_n$ is called the growth bound of the semigroup.

We are now ready to discuss iterative learning schemes for infinite dimensional linear systems. Given an admissible tracking task $y_d(t)$ defined over $[0, t_f]$, we consider the following ILC proportional (P) and differential (D) schemes to handle the iterative error function $e_k(t) = y_d(t) - y_k(t)$:

$$u_{k+1}(t) = u_k(t) + \Gamma e_k(t), \tag{5}$$

$$u_{k+1}(t) = u_k(t) + \Phi\dot{e}_k(t), \tag{6}$$

where k represents the iteration index, Γ and Φ are the learning factors (kernels) which are to be determined to generate a sequence $y_k(t)$ converging to the desired trajectory $y_d(t)$. Motivated by the ILC schemes for SISO systems in [6], we study the convergence conditions of these iteration schemes for infinite dimensional linear systems.

We consider an admissible tracking task $y_d(t)$ over $[0, t_f]$ and an infinite dimensional linear system (1), where A is an infinitesimal generator of a C_0-semigroup. We obtain three convergence conditions in terms of both the P-type and D-type ILC schemes in the rest of this section.

Theorem 1 (P-type): Under the identical initialization condition (IIC) $x_k(0,\xi) = x(0,\xi)$, the proportional type (P-type) ILC scheme (5) generates a convergent sequence y_k in $\mathcal{Y}$ in the sense of the following norm, $\|\cdot\|_{\mathcal{Y},s} = \sup_{t\in[0,t_f]}\|\cdot\|_{\mathcal{Y}}$, if the learning parameter Γ satisfies

$$\|I - D\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} + \|C\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} w(\|\Gamma\|) < 1, \tag{7}$$

where

$$\begin{aligned} w(\|\Gamma\|) &= \|B\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \int_0^{t_f} M_\omega \exp(\omega(t_f - \tau))d\tau \\ &= \begin{cases} M_\omega\|B\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})}\dfrac{\exp(\omega t_f) - 1}{\omega}, & \omega \neq 0, \\ M_\omega\|B\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} t_f, & \omega = 0. \end{cases} \end{aligned}$$

Proof: We first derive the tracking error at each iteration,

$$\begin{aligned} & y_d - y_{k+1} \\ &= y_d - Cx_{k+1} - Du_{k+1} \\ &= y_d - Cx_k - D(u_k + \Gamma e_k) - C(x_{k+1} - x_k) \\ &= (I - D\Gamma)e_k - C(x_{k+1} - x_k) \end{aligned} \tag{8}$$

where I is an identity operator over $\mathcal{Y}$. Based on (2), we can obtain the term $(x_{k+1} - x_k)$ in (8),

$$\begin{aligned} x_{k+1} - x_k &= \int_0^t \mathbb{T}(t-\tau) B \left[u_{k+1}(\tau) - u_k(\tau)\right] d\tau \\ &= \int_0^t \mathbb{T}(t-\tau) B \Gamma e_k(\tau) d\tau \end{aligned} \tag{9}$$

where we have used the identical initialization condition. Then, we estimate the bound of the $\|\cdot\|_{\mathcal{X}}$-norm

$$\begin{aligned} &\|x_{k+1} - x_k\|_{\mathcal{X}} \\ &= \left\| \int_0^{t_f} \mathbb{T}(t-\tau) B \Gamma e_k(\tau) d\tau \right\|_{\mathcal{X}} \\ &\leq \int_0^{t_f} \|\mathbb{T}(t-\tau) B \Gamma e_k(\tau)\|_{\mathcal{X}} \, d\tau \\ &\leq \|B\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \int_0^{t_f} \|\mathbb{T}(t-\tau)\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \|e_k(\tau)\|_{\mathcal{Y}} d\tau \\ &\leq \|B\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \int_0^{t_f} M_\omega \exp(\omega(t_f - \tau)) d\tau \|e_k\|_{\mathcal{Y},s}, \end{aligned} \tag{10}$$

where we have used the growth bound estimate (4). Therefore, we compute $\|\cdot\|_{\mathcal{Y},s}$ of (8) and use the triangular inequality to obtain

$$\begin{aligned} &\|y_d - y_{k+1}\|_{\mathcal{Y},s} \\ &\leq \|(I - D\Gamma) e_k\|_{\mathcal{Y},s} + \|C\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \|x_{k+1} - x_k\|_{\mathcal{X},s} \\ &\leq \|I - D\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} \|e_k\|_{\mathcal{Y},s} + \|C\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} w(\|\Gamma\|) \|e_k\|_{\mathcal{Y},s}. \end{aligned} \tag{11}$$

If we choose the learning parameter Γ to satisfy $\|I - D\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} + \|C\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} w(\|\Gamma\|) < 1$, we obtain the contractive mapping condition $\|e_{k+1}\|_{\mathcal{Y},s} < \|e_k\|_{\mathcal{Y},s}$. Thus, $\lim_{k\to\infty} e_k = 0$, i.e., $\lim_{k\to\infty} y_k = y_d$. ■

Remark 1: In Theorem 1, condition (7) is a nonlinear inequality, which is usually difficult to solve. Alternatively, we can choose the learning kernel Γ such that $\|I - D\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} < \gamma < 1$. Then, there exists a positive constant δ such that $\gamma + \delta < 1$. It is easy to verify that over a subinterval $[0, t_1] \subset [0, t_f]$, where t_1 is given by

$$t_1 \leq \begin{cases} \dfrac{1}{\omega} \ln \left[\dfrac{\delta \omega}{M_\omega \|B\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \|C\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}} + 1 \right], & \omega \neq 0, \\ \dfrac{\delta}{M_\omega \|B\Gamma\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \|C\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}}, & \omega = 0, \end{cases}$$

we obtain $\|e_{k+1}\|_{\mathcal{Y},s} \leq (\gamma + \delta) \|e_k\|_{\mathcal{Y},s} < \|e_k\|_{\mathcal{Y},s}$, which guarantees a geometric convergence of the output tracking error sequence.

Theorem 2 (D-type for bounded A): Assuming that $D = 0$ and A is bounded, under the identical initialization conditions (IICs) $x_k(0,\xi) = x(0,\xi)$ and $y_k(0) = y_d(0)$, the differential type (D-type) ILC scheme (6) generates a convergent sequence y_k in $\mathcal{Y}$ in the sense of the following norm $\|\cdot\|_{\mathcal{Y},s} = \sup_{t\in[0,t_f]} \|\cdot\|_{\mathcal{Y}}$, if the learning parameter Φ satisfies

$$\|I - CB\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} + \|CA\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} w(\|\Phi\|) < 1, \tag{12}$$

where

$$\begin{aligned} w(\|\Phi\|) &= \|B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \int_0^{t_f} M_\omega \exp(\omega(t_f - \tau)) d\tau \\ &= \begin{cases} M_\omega \|B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \dfrac{\exp(\omega t_f) - 1}{\omega}, & \omega \neq 0, \\ M_\omega \|B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} t_f, & \omega = 0. \end{cases} \end{aligned}$$

Proof: We compute the time derivative of the tracking error at each iteration

$$\begin{aligned} \dot{e}_{k+1} &= \dot{y}_d - C\dot{x}_{k+1} \\ &= \dot{y}_d - CAx_{k+1} - CB(u_k + \Phi \dot{e}_k) \\ &= \dot{y}_d - C\dot{x}_k - CB\Phi\dot{e}_k - CA\left(x_{k+1} - x_k\right) \\ &= (I - CB\Phi)\dot{e}_k - CA\left(x_{k+1} - x_k\right), \end{aligned} \tag{13}$$

where the state difference between two adjacent iterations is given by (we use the the IICs $x_k(0,\xi) = x(0,\xi)$),

$$\begin{aligned} &x_{k+1}(t) - x_k(t) \\ &= \int_0^t \mathbb{T}(t-\tau) B \left[u_{k+1}(\tau) - u_k(\tau)\right] d\tau \\ &= \int_0^t \mathbb{T}(t-\tau) B \Phi \dot{e}_k(\tau) d\tau. \end{aligned} \tag{14}$$

Then, we can obtain the bound estimate in terms of the $\|\cdot\|_{\mathcal{X}}$-norm as

$$\begin{aligned} &\|x_{k+1} - x_k\|_{\mathcal{X}} \\ &= \left\| \int_0^{t_f} \mathbb{T}(t-\tau) B \Phi \dot{e}_k(\tau) d\tau \right\|_{\mathcal{X}} \\ &\leq \int_0^{t_f} \|\mathbb{T}(t-\tau) B \Phi \dot{e}_k(\tau)\|_{\mathcal{X}} \, d\tau \\ &\leq \int_0^{t_f} \|\mathbb{T}(t_f-\tau)\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} \|B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \, d\tau \|\dot{e}_k\|_{\mathcal{Y},s} \\ &\leq \|B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \int_0^{t_f} M_\omega \exp(\omega(t_f - \tau)) d\tau \|\dot{e}_k\|_{\mathcal{Y},s} \end{aligned} \tag{15}$$

We compute the $\|\cdot\|_{\mathcal{Y},s}$-norm of (13) and use the triangular inequality to obtain

$$\begin{aligned} \|\dot{e}_{k+1}\|_{\mathcal{Y},s} &\leq \|(I - CB\Phi)\dot{e}_k\|_{\mathcal{Y},s} \\ &\quad + \|CA\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \|x_{k+1} - x_k\|_{\mathcal{X},s} \\ &\leq \|I - CB\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} \|\dot{e}_k\|_{\mathcal{Y},s} \\ &\quad + \|CA\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} w(\|\Phi\|) \|\dot{e}_k\|_{\mathcal{Y},s}. \end{aligned} \tag{16}$$

If we choose the learning parameter Φ to satisfy $\|I - CB\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} + \|CA\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} w(\|\Phi\|) < 1$, we obtain the contractive mapping condition $\|\dot{e}_{k+1}\|_{\mathcal{Y},s} < \|\dot{e}_k\|_{\mathcal{Y},s}$. Thus, $\lim_{k\to\infty} \dot{e}_k = 0$. Since $e_k(0) = 0$ by the IIC assumption, we have $0 \leq \lim_{k\to\infty} \|e_k\|_{\mathcal{Y},s} \leq \|e_k(0)\|_{\mathcal{Y}} + \lim_{k\to\infty} \int_0^t \|\dot{e}_k(\tau)\| \, d\tau = 0$, $\forall t \in [0, t_f]$, i.e., $\lim_{k\to\infty} y_k = y_d$. ■

Remark 2: In Theorem 2, condition (12) is a nonlinear inequality, which is usually difficult to solve. Alternatively, we can choose the learning kernel Φ such that $\|I - CB\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} < \gamma < 1$. Then, there exists a positive constant δ such that $\gamma + \delta < 1$. It is easy to verify that over the subinterval $[0, t_1] \subset [0, t_f]$, where t_1 is given by

$$t_1 \leq \begin{cases} \dfrac{1}{\omega} \ln \left[\dfrac{\delta \omega}{M_\omega \|B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \|CA\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}} + 1 \right], & \omega \neq 0, \\ \dfrac{\delta}{M_\omega \|B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} \|CA\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})}}, & \omega = 0, \end{cases}$$

we obtain $\|\dot{e}_{k+1}\|_{\mathcal{Y},s} \leq (\gamma + \delta) \|\dot{e}_k\|_{\mathcal{Y},s} < \|\dot{e}_k\|_{\mathcal{Y},s}$, which guarantees a geometric convergence of the output tracking error sequence by taking into account the IIC assumptions.

Usually the operator A is unbounded. In this case, Theorem 2 is not applicable since the convergence condition is expressed in terms of the norm of the operator A.

Theorem 3 (D-type for unbounded A): Assuming that $D = 0$ and A is unbounded, under the identical initialization conditions (IICs) $x_k(0, \xi) = x(0, \xi)$ and $y_k(0) = y_d(0)$, the differential type (D-type) ILC scheme (6) generates a convergent sequence y_k in $\mathcal{Y}$ in the sense of the following norm $\| \cdot \|_{\mathcal{Y},s} = \sup_{t \in [0,t_f]} \| \cdot \|_{\mathcal{Y}}$, if the learning parameter Φ satisfies

$$\begin{aligned} &\|I - C\mathbb{T}(0)B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} \\ &+ \int_0^{t_f} \left\| C \frac{d\mathbb{T}(t_f - \tau)}{dt} B\Phi \right\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} d\tau < 1. \end{aligned} \tag{17}$$

Proof: We compute the time derivative of the tracking error at each iteration

$$\begin{aligned} \dot{e}_{k+1} &= \dot{y}_d - C\dot{x}_{k+1} \\ &= \dot{y}_d - C\frac{d}{dt}\left[\mathbb{T}(t)x_0 + \int_0^t \mathbb{T}(t-\tau)Bu_{k+1}(\tau)d\tau\right] \\ &= \dot{y}_d - C\frac{d}{dt}\mathbb{T}(t)x_0 + \mathbb{T}(0)Bu_{k+1}(t) \\ &\quad - C\int_0^t \frac{d}{dt}\mathbb{T}(t-\tau)Bu_{k+1}(\tau)d\tau. \end{aligned} \tag{18}$$

We substitute the D-type ILC scheme (6) into (18) to obtain

$$\begin{aligned} \dot{e}_{k+1}(t) &= \dot{y}_d(t) - C\frac{d}{dt}\mathbb{T}(t)x_0 + \mathbb{T}(0)Bu_k(t) \\ &\quad - C\int_0^t \frac{d}{dt}\mathbb{T}(t-\tau)Bu_k(\tau)d\tau \\ &\quad + \mathbb{T}(0)B\Phi\dot{e}_k(t) \\ &\quad - C\int_0^t \frac{d}{dt}\mathbb{T}(t-\tau)B\Phi\dot{e}_k(\tau)d\tau \\ &= \left[I - \mathbb{T}(0)B\Phi\right]\dot{e}_k(t) \\ &\quad - C\int_0^t \frac{d}{dt}\mathbb{T}(t-\tau)B\Phi\dot{e}_k(\tau)d\tau. \end{aligned} \tag{19}$$

We compute the $\| \cdot \|_{\mathcal{Y},s}$-norm of (19) to obtain

$$\begin{aligned} \|\dot{e}_{k+1}\|_{\mathcal{Y},s} &\leq \|I - \mathbb{T}(0)B\Phi\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} \|\dot{e}_k\|_{\mathcal{Y},s} \\ &+ \int_0^{t_f} \left\| C\frac{d}{dt}\mathbb{T}(t_f - \tau)B\Phi \right\|_{\mathcal{L}(\mathcal{Y},\mathcal{Y})} d\tau \|\dot{e}_k\|_{\mathcal{Y},s}. \end{aligned} \tag{20}$$

Therefore, if the learning parameter Φ satisfies the condition (17), we obtain the contractive mapping condition $\|\dot{e}_{k+1}\|_{\mathcal{Y},s} < \|\dot{e}_k\|_{\mathcal{Y},s}$. Thus, by following the same arguments presented in the proof of Theorem 2, we obtain $\lim_{k\to\infty} y_k = y_d$. ∎

III. Numerical Illustration

We consider a metal bar of length one $0 \leq \xi \leq 1$ that can be heated along its length by a given actuating distribution function $f(\xi)$. The governing equation is given by

$$\begin{cases} \dfrac{\partial x}{\partial t} = \dfrac{\partial^2 x}{\partial \xi^2} + f(\xi)u(t), \\ x(0, \xi) = x_0(\xi), \\ \dfrac{\partial x}{\partial \xi}(t, 0) = \dfrac{\partial x}{\partial \xi}(t, 1) = 0, \end{cases} \tag{21}$$

where $x(t, \xi)$ represents the temperature at position ξ at time t, $x_0(\xi)$ the initial temperature profile, $f(\xi) = \xi(\xi - 1)$ the actuating distribution function, and $u(t)$ the scalar control. By measuring the temperature at the right end, we define the system output as

$$y(t) = x(t, 1) \approx \int_0^1 \mathbf{1}_{[1-\epsilon,1]}(\xi)x(t, \xi)d\xi. \tag{22}$$

where ϵ is a small positive constant and

$$\mathbf{1}_{[\xi_0-\epsilon,\xi_0+\epsilon]}(\xi) = \begin{cases} 1, & \xi \in [\xi_0 - \epsilon, \xi_0 + \epsilon], \\ 0, & \text{elsewhere}. \end{cases}$$

To rewrite this PDE system as an abstract linear system, we choose the state space as $\mathcal{X} = L^2(0, 1) = \left\{ x(t) | \int_0^1 x^2(t, \xi)d\xi < \infty, \forall t \in [0, t_f] \right\}$ and the state as $x(t, \cdot) = \{x(t, \xi), 0 \leq \xi \leq 1\}$. We introduce the system operator A on $\mathcal{X}$ to be $Ax = \frac{d^2x}{d\xi^2}$ with the domain given by $\mathrm{dom}(A) = \{x \in H^1(0, 1), \frac{\partial x}{\partial \xi}(t, 0) = \frac{\partial x}{\partial \xi}(t, 1) = 0\}$, where $H^1(0, 1)$ is defined by $H^1(0, 1) = \left\{ x | x \in L^2(0, 1) \text{ and } \frac{\partial x}{\partial \xi} \in L^2(0, 1) \right\}$. The input matrix B is defined as $B = f(\xi)$ and the output operator as $Cx = x(t, 1)$.

We first use separation of variables [20] to write

$$x(t, \xi) = T(t)X(\xi) \tag{23}$$

for the homogenous system (21) (with $u(t) = 0$) to obtain the Sturm-Liouville equation which is a two boundary value problem (TBVP)

$$AX + \lambda X = 0 \Rightarrow \begin{cases} X''(\xi) + \lambda X(\xi) = 0, \\ X'(0) = X'(1) = 0. \end{cases} \tag{24}$$

The solution of (24) is

$$X_0(\xi) = 1, \quad \lambda_0 = 0, \tag{25}$$

$$X_n(\xi) = \sqrt{2}\cos(n\pi\xi), \quad \lambda_n = n^2\pi^2, \ n > 0. \tag{26}$$

Then, based on (23), the solution representation becomes

$$x(t, \xi) = T_0(t) + \sum_{n=1}^{\infty} T_n(t)X_n(\xi), \tag{27}$$

which satisfies

$$\begin{cases} \sum_{n=0}^{\infty} \left(\frac{dT_n(t)}{dt} + \lambda_n T_n(t) \right) X_n(\xi) = f(\xi)u(t), \\ \sum_{n=0}^{\infty} T_n(0) X_n(\xi) = x_0(\xi) \Rightarrow T_n(0) = \int_0^1 x_0(\xi) X_n(\xi) d\xi. \end{cases}$$

We can now expand $f(\xi)$ in terms of $\{X_n\}_{n=0}^{\infty}$ as $f(\xi) = \sum_{n=0}^{\infty} f_n X_n(\xi)$, $f_n = \int_0^1 f(\xi) X_n(\xi) d\xi$, in order to obtain

$$\frac{dT_n(t)}{dt} + \lambda_n T_n(t) = f_n u(t), T_n(0) = \int_0^1 x_0(\xi) X_n(\xi) d\xi.$$

The solutions for each component $T_n(t)$ are readily obtained as $(n \geq 0)$

$$\begin{aligned} T_n(t) &= \exp(-\lambda_n t) \int_0^1 x_0(\zeta) X_n(\zeta) d\zeta \\ &+ \int_0^t \exp(-\lambda_n (t-\tau)) u(\tau) d\tau \int_0^1 f(\zeta) X_n(\zeta) d\zeta. \end{aligned} \tag{28}$$

Therefore, the solution of the PDE system (21) is given by

$$x(t,\xi) = \int_0^1 g(t,\xi,\zeta) x_0(\zeta) d\zeta + \int_0^t \int_0^1 g(t-\tau,\xi,\zeta) f(\zeta) u(\tau) d\zeta d\tau,$$

where $g(t,\xi,\zeta) = 1 + \sum_{n=1}^{\infty} 2\exp(-n^2\pi^2 t)\cos(n\pi\zeta)\cos(n\pi\xi)$. We consider the D-type iterative learning control scheme (6) in this numerical study. The time derivative of the error is

$$\begin{aligned} \dot{e}_{k+1}(t) &= \left[1 - \Phi \int_0^1 g(0,1,\zeta) f(\zeta) d\zeta \right] \dot{e}_k(t) \\ &- \int_0^t \int_0^1 \frac{\partial g}{\partial t}(t-\tau,1,\zeta) f(\zeta) \Phi \dot{e}_k(\tau) d\zeta d\tau. \end{aligned} \tag{29}$$

By using the convergence condition (17) in Theorem 3, we can obtain an inequality condition for the learning gain Φ. Recalling that $f(\xi) = \xi(\xi - 1)$ and $\cos(n\pi\xi)|_{\xi=1} = \cos(n\pi) = (-1)^n$, $g(0,1,\zeta) = 1 + \sum_{n=1}^{\infty} 2(-1)^n \cos(n\pi\zeta)$ and $\int_0^1 \cos(n\pi\zeta) f(\zeta) d\zeta = \frac{1+(-1)^n}{n^2\pi^2}$, $n = 1, 2, \ldots$, we can write the time derivative of $e_{k+1}(t)$ as

$$\begin{aligned} \dot{e}_{k+1}(t) &= \left[1 - \Phi \int_0^1 f(\zeta) d\zeta - \Phi \sum_{n=1}^{\infty} 2 \frac{1+(-1)^n}{n^2\pi^2} \right] \dot{e}_k(t) \\ &+ \Phi \sum_{n=1}^{\infty} [(-1)^n + 1] \int_0^t 2\exp(-n^2\pi^2(t-\tau)) \dot{e}_k(\tau) d\tau. \end{aligned}$$

We compute the norm $|\cdot|_s$ of $\dot{e}_{k+1}$ to obtain

$$\begin{aligned} |\dot{e}_{k+1}|_s &\leq \left| 1 - \Phi \int_0^1 f(\zeta) d\zeta - \Phi \sum_{n=1}^{\infty} 2 \frac{1+(-1)^n}{n^2\pi^2} \right| |\dot{e}_k|_s \\ &+ \left| \Phi \sum_{n=1}^{\infty} 4 \int_0^t \exp(-4n^2\pi^2(t-\tau)) d\tau \right|_s |\dot{e}_k|_s \\ &= \left| 1 + \frac{\Phi}{6} - \frac{\Phi}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right| |\dot{e}_k|_s \\ &+ \left| \Phi \sum_{n=1}^{\infty} \frac{1 - \exp(-4n^2\pi^2 t)}{n^2\pi^2} \right|_s |\dot{e}_k|_s. \end{aligned}$$

By noting that $\frac{1}{n(n+1)} < \frac{1}{n^2} < \frac{1}{n(n-1)}$, $n \geq 2$, we can then obtain

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n-1} \right),$$

i.e., $1 < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{3}{2}$. Therefore, the the norm $|\cdot|_s$ of $\dot{e}_{k+1}$ becomes

$$\begin{aligned} |\dot{e}_{k+1}|_s &\leq \left| 1 + \left(\frac{1}{6} - \frac{\sum_{n=1}^{\infty} \frac{1}{n^2}}{2\pi^2} \right) \Phi \right| |\dot{e}_k|_s \\ &+ \left| \Phi \sum_{n=1}^{\infty} \frac{1 - \exp(-4n^2\pi^2 t_f)}{n^2\pi^2} \right| |\dot{e}_k|_s. \end{aligned} \tag{30}$$

We note that $\frac{1}{6} - \frac{\sum_{n=1}^{\infty} \frac{1}{n^2}}{2\pi^2} > \frac{1}{6} - \frac{3}{4\pi^2} = \frac{2\pi^2 - 9}{12\pi^2} > 0$, and based on the inequality $\left| 1 + \left(\frac{1}{6} - \frac{\sum_{n=1}^{\infty} \frac{1}{n^2}}{2\pi^2} \right) \Phi \right| < 1$, we obtain

$$\left(\frac{12\pi^2}{3 - \pi^2} < \frac{12\pi^2}{3\sum_{n=1}^{\infty} \frac{1}{n^2} - \pi^2} = \right) \frac{2}{\frac{\sum_{n=1}^{\infty} \frac{1}{n^2}}{2\pi^2} - \frac{1}{6}} < \Phi < 0,$$

i.e., $-17.2405 < \Phi < 0$. Finally, if we choose $\Phi = -5$, then there exists $t_1 < t_f$ such that

$$\left| \frac{1}{6} + 5 \frac{\sum_{n=1}^{\infty} \frac{1}{n^2}}{2\pi^2} \right| + 5 \left| \sum_{n=1}^{\infty} \frac{1 - \exp(-4n^2\pi^2 t_1)}{n^2\pi^2} \right| < 1. \tag{31}$$

Thus, the sequence $\dot{e}_k(t)$ is convergent. By taking into account the identical initialization condition, i.e., $e_k(0) = y_d(0) - y_k(0) = 0$, we can then conclude that the error sequence $e_k(t)$ is also convergent. We numerically solve the PDE system using the finite element method [21]. In Fig. 1, the evolution of the PDE system is shown for a random input signal labeled as "iteration 1" in Fig. 2. Then, we use the D-type ILC scheme (6) with $\Phi = -5$ to iterate the simulation and update the control input. After four iterations, the control input converges (shown in Fig. 2) and the output (right boundary measurement) converges to the desired tracking trajectory (shown in Fig. 3). With the input function obtained at the fourth iteration, we simulate the PDE system and the evolution is shown in Fig. 4.

IV. Conclusions

In this paper, we have extended the ILC scheme to distributed parameter systems governed by parabolic PDEs. We have proposed a general approach with the potential of being applied to PDE-constrained optimization problems. We have discussed both the P-type and D-type ILC schemes for a distributed parameter system formulated as a general linear system on a Hilbert space, in which the system operator generates a C_0-semigroup. Three learning parameter conditions have been obtained to guarantee the convergence of the P-type and D-type ILC schemes. The conditions do not require analytical solutions but bounds in appropriate norm spaces for the system operators and semigroups. Numerical simulations have been carried out for ILC applied to a 1D heat conduction problem. The semigroup analysis has been considered for the synthesis of the ILC

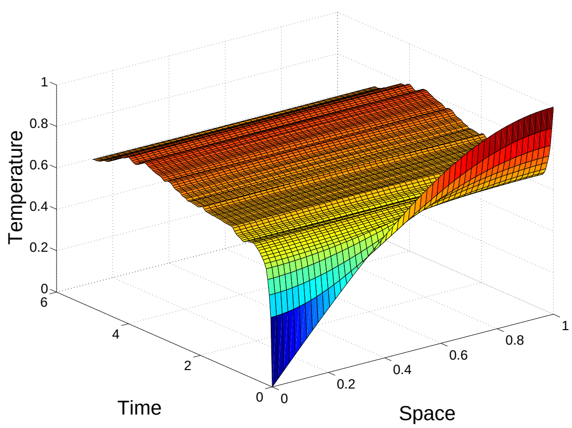

Fig. 1. System evolution for random input (first iteration).

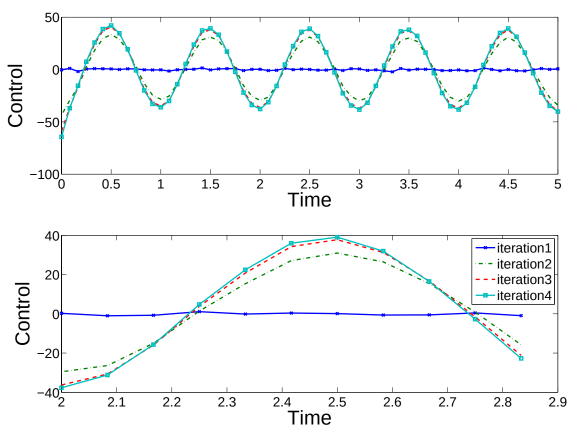

Fig. 2. Iterations for the control input function.

parameters. The simulation results show the effectiveness of the proposed ILC scheme. Future research topics include the study of the potential of combining numerical discretization and finite dimensional ILC to define a general ILC synthesis framework for distributed parameter systems. Additionally, technical challenges remain in extending the ILC approach to infinite dimensional systems with boundary observations and point controls that involve unbounded operators.

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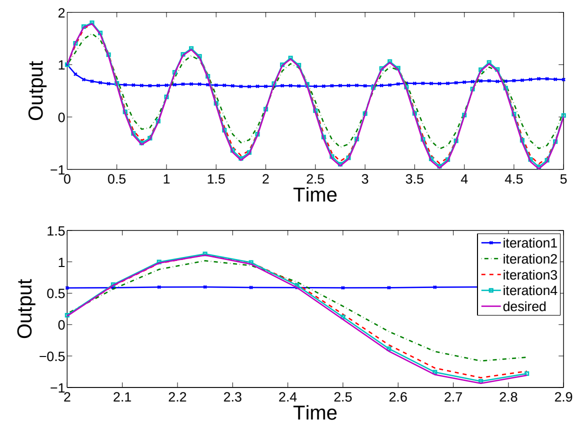

Fig. 3. Iterations for the output function.

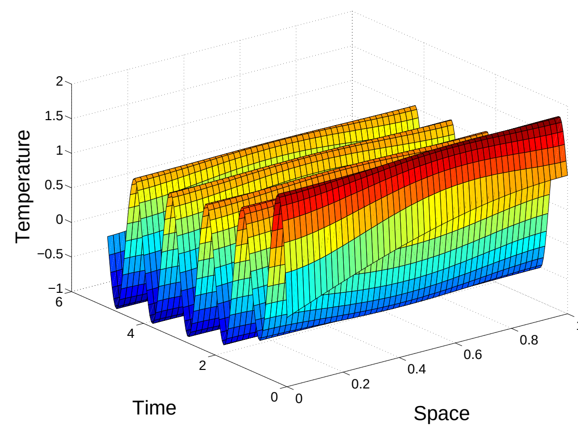

Fig. 4. System evolution for control input obtained after four iterations.

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