A Closed-Form Feedback Controller for Stabilization of Magnetohydrodynamic Channel Flow

Rafael Vazquez, Eugenio Schuster and Miroslav Krstic

Abstract— **We present a PDE boundary controller that stabilizes the velocity, pressure, and electromagnetic fields in a magnetohydrodynamic (MHD) channel flow, also known as Hartmann flow, a benchmark model for applications such as cooling systems, hypersonic flight and propulsion. This flow is characterized by an electrically conducting fluid moving between parallel plates in the presence of an externally imposed transverse magnetic field. The system is described by the inductionless MHD equations, a combination of the Navier-Stokes equations and a Poisson equation for the electric potential under the so-called MHD approximation in a low magnetic Reynolds number regime, and is unstable for large Reynolds numbers. Our control design needs actuation of velocity and the electric potential at only one of the walls. The backstepping method for stabilization of parabolic PDEs is applied to the velocity field system written in some appropriate coordinates; this system is very similar to the Orr-Sommerfeld-Squire system of PDE's and presents the same difficulties. Thus we use actuation not only to guarantee stability but also to decouple the system in order to prevent transients. Control gains are computed solving linear hyperbolic PDEs—a much simpler task than, for instance, solving nonlinear Riccati equations. Stabilization of non-discretized 3-D MHD channel flow has so far been an open problem.**

I. Introduction

In this paper we consider an incompressible MHD channel flow, also known as the Hartmann flow, a benchmark model for applications such as cooling systems (computer systems, fusion reactors), hypersonic flight and propulsion. In this flow, an electrically conducting fluid moves between parallel plates and is affected by an imposed transverse magnetic field. When a conducting fluid moves in the presence of a magnetic field, it produces an electric field and subsequently an electric current. The interaction between this created electric current and the imposed magnetic field originates a body force, called the Lorentz force, which acts on the fluid itself. The velocity and electromagnetic fields are mathematically described by the MHD equations [16], which are the Navier-Stokes equation coupled with the Maxwell equations.

For non-conducting fluids, channel flow is a benchmark for flow control, frequently cited as a paradigm for transition to turbulence [19]. There are many results in channel flow stabilization, for instance, using optimal control [12], backstepping [10], [26], spectral decomposition/pole placement [7], [24], Lyapunov design/passivity [1], [3], or nonlinear model reduction/in-domain actuation [2].

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The area of conducting fluids moving in magnetic fields, even though rich in applications, has only been recently considered and is under development. There are some recent results in stabilization of magnetohydrodynamic flows, for instance using nonlinear model reduction [4], open-loop control [8] and optimal control [11]. Applications include drag reduction [17], mixing enhancement for cooling systems [18], or estimation of velocity, pressure and electromagnetic fields [27]. Some experimental results are available as well, showing that control of such flows is technologically feasible; actuators consist of magnets and electrodes [9], [17], [23]. Mathematical studies of controllability of magnetohydrodynamic flows have been done, though they do not provide explicit controllers [6], [22].

This paper is based on our previous work for stabilization of the velocity field in a 3-D channel flow [10]. Our controller is designed for the continuum MHD model. Since the system is spatially invariant [5], control synthesis is done in the wave number space after application of a Fourier transform. Large wave numbers are found to be stable and left uncontrolled whereas for small wave numbers control is used. For these wave numbers, control is used to put the system in a strict-feedback form; this is necessary for application of the backstepping method for stabilization of parabolic PDE's [21]. Writing the velocity field in some appropriate coordinates, the resulting system is very similar to the Orr-Sommerfeld-Squire system of PDE's for non-conducting fluids and presents the same difficulties (non-normality leading to a large transient growth mechanism [13], [19]). Thus, applying the same ideas as in [10], we use backstepping not only to guarantee stability but also to decouple the system in order to prevent transients. The control gains are computed solving linear hyperbolic PDEs—a much simpler task than, for instance, solving nonlinear Riccati equations. Actuation of velocity and electric potential is done at only one of the channel walls. Full state knowledge is assumed, but the controller can be combined with a dual observer for MHD channel flow [27] to obtain an output feedback controller.

The paper is organized as follows. Section II introduces the governing equations of our system. The equilibrium profile is presented in Section III and the linearized plant in wave number space introduced in Section IV. Section V presents the design of the control laws to guarantee stability of the closed-loop system and finishes stating our main result.

II. Model

Consider an incompressible conducting fluid enclosed between two plates, separated by a distance L, under the influence of a pressure gradient ∇P and a magnetic field B_0 normal to the walls, as shown in Figure 1. Under the

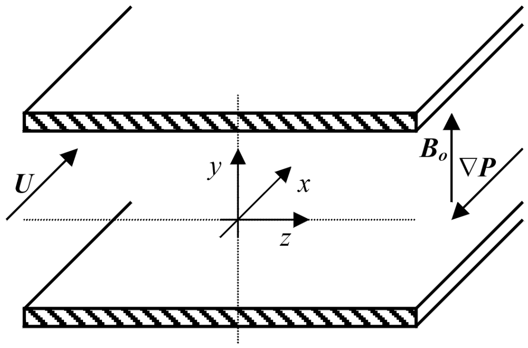

Fig. 1. Hartmann Flow.

assumption of a very small magnetic Reynolds number

$$Re_M = \nu\rho\sigma U_0 L \ll 1, \tag{1}$$

where ν is the viscosity of the fluid, ρ the density of the fluid, σ the conductivity of the fluid, and U_0 the reference velocity (maximum velocity of the equilibrium profile), the dynamics of the magnetic field can be neglected and the dimensionless velocity and electric potential field is governed by the inductionless MHD equations [15].

We set nondimensional coordinates (x, y, z), where x is the streamwise direction (parallel to the pressure gradient), y the wall normal direction (parallel to the magnetic field) and z the spanwise direction, so that $(x, y, z) \in (-\infty, \infty) \times [0, 1] \times (-\infty, \infty)$[1]. The governing equations are

$$\begin{aligned} U_t &= \frac{\triangle U}{Re} - UU_x - VU_y - WU_z - P_x + N\phi_z \\ &\quad -NU\,, && (2) \\ V_t &= \frac{\triangle V}{Re} - UV_x - VV_y - WV_z - P_y\,, && (3) \\ W_t &= \frac{\triangle W}{Re} - UW_x - VW_y - WW_z - P_z - N\phi_x \\ &\quad -NW\,, && (4) \\ \triangle\phi &= U_z - W_x\,, && (5) \end{aligned}$$

where U, V and W denote, respectively, the streamwise, wall-normal and spanwise velocities, P the pressure, ϕ the electric potential, $Re = \frac{U_0 L}{\nu}$ is the Reynolds number and $N = \frac{\sigma L B_0^2}{\rho U_0}$ the Stuart number. Since the fluid is incompressible, the continuity equation is verified

$$U_x + V_y + W_z = 0\,. \tag{6}$$

The boundary conditions for the velocity field are

$$\begin{aligned} U(t, x, 0, z) &= 0,\ U(t, x, 1, z) = U_c(t, x, z), && (7) \\ V(t, x, 0, z) &= 0,\ V(t, x, 1, z) = V_c(t, x, z), && (8) \\ W(t, x, 0, z) &= 0,\ W(t, x, 1, z) = W_c(t, x, z), && (9) \end{aligned}$$

where $U_c(t, x, z)$, $V_c(t, x, z)$ and $W_c(t, x, z)$ denote, respectively, the actuators for streamwise, wall-normal and spanwise velocity in the upper wall. Assuming perfectly conducting walls, the electric potential must verify

$$\phi(t, x, 0, z) = 0,\ \phi(t, x, 1, z) = \Phi_c(t, x, z), \tag{10}$$

where $\Phi_c(t, x, z)$ is the imposed potential (electromagnetic actuation) in the upper wall. The nondimensional electric current, $j(t, x, y, z)$, a vector field that is computed from the electric potential and velocity fields as follows,

$$\begin{aligned} j^x(t, x, y, z) &= -\phi_x - W, && (11) \\ j^y(t, x, y, z) &= -\phi_y, && (12) \\ j^z(t, x, y, z) &= -\phi_z + U, && (13) \end{aligned}$$

where j^x, j^y, and j^z denote the components of j.

We assume that all actuators can be independently actuated for every $(x, z) \in \mathbb{R}^2$. Note that no actuation is done inside the channel or at the bottom wall.

III. Equilibrium profile

The equilibrium profile for system (2)–(5) with no control can be calculated following the same steps that yield the Poiseuille solution for Navier-Stokes channel flow. Thus, we assume a steady solution with only one nonzero nondimensional velocity component, $U^e(y)$, that depends only on the y coordinate. Substituting $U^e(y)$ in equation (2), one finds that it verifies the following equation,

$$0 = \frac{U^e_{yy}(y)}{Re} - P^e_x - NU^e(y)\,, \tag{14}$$

whose nondimensional solution is, setting P^e such that the maximum velocity (centerline velocity) is unity,

$$\begin{aligned} U^e(y) &= \frac{\sinh(H(1-y)) - \sinh H + \sinh(Hy)}{2\sinh H/2 - \sinh H}, && (15) \\ V^e &= W^e = \phi^e = 0, && (16) \\ P^e &= \frac{N \sinh H}{2\sinh H/2 - \sinh H} x, && (17) \\ j^{xe} &= j^{ye} = 0, j^{ze} = U^e(y). && (18) \end{aligned}$$

where $H = \sqrt{ReN} = B_0 L \sqrt{\frac{\sigma}{\rho\nu}}$ is the Hartmann number. In Fig. III(left) we show $U^e(y)$ for different values of H. Since the equilibrium profile is nondimensional the centerline velocity is always 1. For $H = 0$ the classic parabolic Poiseuille profile is recovered. In Fig. III(right) we show $U^e_y(y)$, proportional to shear stress, whose maximum is reached at the boundaries and grows with H.

IV. The Plant in Wave Number Space

Define the fluctuation variables

$$\begin{aligned} u(t, x, y) &= U(t, x, y) - U^e(y), && (19) \\ p(t, x, y) &= P(t, x, y) - P^e(y) && (20) \end{aligned}$$

where $U^e(y)$ and $P^e(y)$ are, respectively, the equilibrium velocity and pressure given in (15) and (17). The linearization of (2)–(4) around the Hartmann equilibrium profile, written in the fluctuation variables (u, V, W, p, ϕ), is

$$\begin{aligned} u_t &= \frac{\triangle u}{Re} - U^e(y)u_x - U^e_y(y)V - p_x + N\phi_z \\ &\quad -Nu\,, && (21) \\ V_t &= \frac{\triangle V}{Re} - U^e(y)V_x - p_y\,, && (22) \\ W_t &= \frac{\triangle W}{Re} - U^e(y)W_x - p_z - N\phi_x - NW\,. && (23) \end{aligned}$$

[1]Our approach can be extended to finite, periodic channels with only some changes; see e.g. [25] for techniques involved.

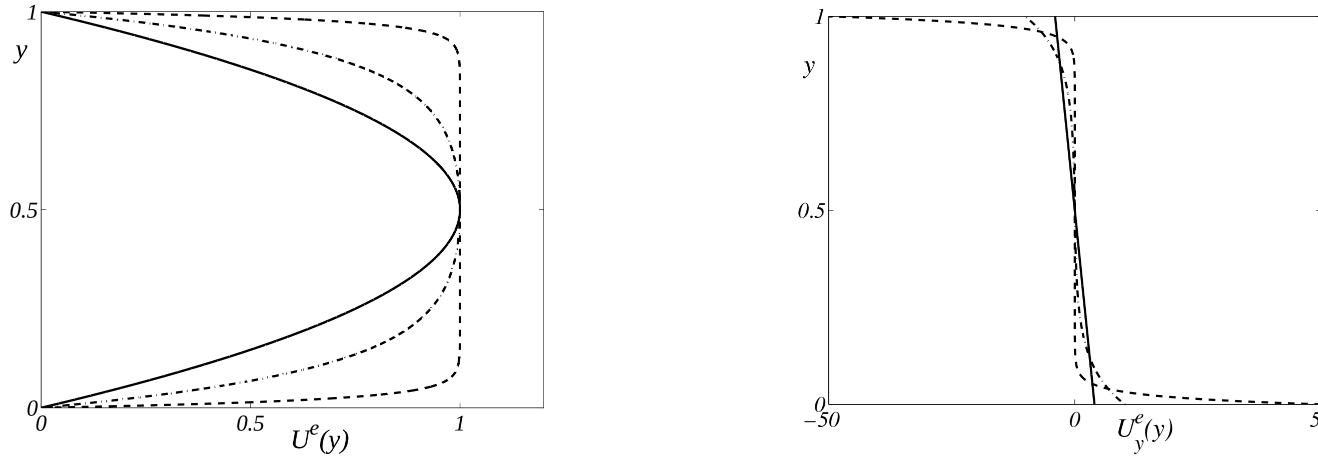

Fig. 2. Streamwise equilibrium velocity $U^e(y)$ (left) and $U_y^e(y)$ (right), for different values of H. Solid, $H = 0$; dash-dotted, $H = 10$; dashed, $H = 50$.

The equation for the potential is

$$\triangle\phi = u_z - W_x\,, \tag{24}$$

and the fluctuation velocity field verifies the continuity equation,

$$u_x + V_y + W_z = 0\,, \tag{25}$$

and the following boundary conditions

$$\begin{aligned} u(t,x,0,z) &= W(t,x,0,z) = V(t,x,0,z) = 0, && (26)\\ u(t,x,1,z) &= U_c(t,x,z), && (27)\\ V(t,x,1,z) &= V_c(t,x,z), && (28)\\ W(t,x,1,z) &= W_c(t,x,z), && (29)\\ \phi(t,x,0,z) &= 0,\ \phi(t,x,1,z) = \Phi_c(t,x,z). && (30) \end{aligned}$$

To guarantee stability, our design task is to design feedback laws U_c, V_c, W_c and Φ_c, so that the origin of the velocity fluctuation system is exponentially stable. Full state knowledge is assumed.

Since the plant is linear and spatially invariant [5], we use a Fourier transform in the x and z coordinates (the spatially invariant directions). The transform pair (direct and inverse transform) is defined as

$$f(k_x,y,k_z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\mathrm{e}^{-2\pi i(k_x x + k_z z)}dz dx, \tag{31}$$

$$f(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x,y,k_z)\mathrm{e}^{2\pi i(k_x x + k_z z)}dk_z dk_x. \tag{32}$$

Note that we use the same symbol f for both the original $f(x,y,z)$ and the image $f(k_x,y,k_z)$. In hydrodynamics k_x and k_z are referred to as the "wave numbers."

The plant equations in wave number space are

$$\begin{aligned} u_t &= \frac{-\alpha^2 u + u_{yy}}{Re} - \beta(y)u - U_y^e(y)V - 2\pi k_x ip \\ &\quad +2\pi k_z iN\phi - Nu, && (33)\\ V_t &= \frac{-\alpha^2 V + V_{yy}}{Re} - \beta(y)V - p_y, && (34)\\ W_t &= \frac{-\alpha^2 W + W_{yy}}{Re} - \beta(y)W - 2\pi k_z ip \\ &\quad -2\pi k_x iN\phi - NW && (35) \end{aligned}$$

where $\alpha^2 = 4\pi^2(k_x^2 + k_z^2)$ and $\beta(y) = 2\pi i k_x U^e(y)$.

The continuity equation in wave number space is expressed as

$$2\pi i k_x u + V_y + 2\pi k_z W = 0, \tag{36}$$

and the equation for the potential is

$$-\alpha^2\phi + \phi_{yy} = 2\pi i\left(k_z u - k_x W\right). \tag{37}$$

The boundary conditions are

$$\begin{aligned} u(t,k_x,0,k_z) &= W(t,k_x,0,k_z) = V(t,k_x,0,k_z) = 0, && (38)\\ u(t,k_x,1,k_z) &= U_c(t,k_x,k_z), && (39)\\ V(t,k_x,1,k_z) &= V_c(t,k_x,k_z), && (40)\\ W(t,k_x,1,k_z) &= W_c(t,k_x,k_z), && (41)\\ \phi(t,k_x,0,k_z) &= 0,\ \phi(t,k_x,1,k_z) = \Phi_c(t,k_x,k_z). && (42) \end{aligned}$$

V. Control Design

We design the controller in wave number space. Note that (33)–(42) are uncoupled for each wave number. Therefore, as in [10], [26], the range $k_x^2 + k_z^2 \le M^2$, which we refer to as the *controlled* wave number range, and the range $k_x^2 + k_z^2 > M^2$, the *uncontrolled* wave number range, can be studied separately. If stability for all wave numbers is established, stability in physical space follows (see [26]). The number M, which will be computed in Section V-B, is a parameter that ensures stability for uncontrolled wave numbers.

We define χ, a *truncating* function, as

$$\chi(k_x,k_z) = \begin{cases} 1, & k_x^2 + k_z^2 \le M^2, \\ 0, & \text{otherwise.} \end{cases} \tag{43}$$

Then, we reflect that we don't use control for large wave numbers by setting

$$\begin{pmatrix} U_c(t,x,z) \\ V_c(t,x,z) \\ W_c(t,x,z) \\ \Phi_c(t,x,z) \end{pmatrix} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} U_c(t,k_x,k_z) \\ V_c(t,k_x,k_z) \\ W_c(t,k_x,k_z) \\ \Phi_c(t,k_x,k_z) \end{pmatrix} \times\chi(k_x,k_z)\mathrm{e}^{2\pi i(k_x x + k_z z)}dk_z dk_x. \tag{44}$$

Next we design stabilizing control laws for small wave numbers and analyze uncontrolled wave numbers.

A. Controlled wave number analysis

Consider $k_x^2+k_z^2\leq M^2$. Then $\chi=1$, so there is control. Using the continuity equation (36) and taking divergence of (33)–(35), a Poisson equation for the pressure is derived,

$$-\alpha^2 p+p_{yy}=-4\pi k_x i U_y^e(y)V+NV_y. \tag{45}$$

Evaluating equation (34) at $y=0$ one finds that

$$\begin{aligned} p_y(k_x,0,k_z) &= \frac{V_{yy}(k_x,0,k_z)}{Re} \\ &= -2\pi i\frac{k_x u_{y0}+k_z W_{y0}}{Re}, \end{aligned} \tag{46}$$

where we use (36) for expressing V_{yy} at the bottom in terms of $u_{y0}=u_y(k_x,0,k_z)$ and $W_{y0}=W_y(k_x,0,k_z)$. Similarly, evaluating equation (34) at $y=1$ we get

$$\begin{aligned} p_y(k_x,1,k_z) &= \frac{V_{yy}(k_x,1,k_z)}{Re}-(V_c)_t-\alpha^2\frac{V_c}{Re} \\ &= -2\pi i\frac{k_x u_{y1}+k_z W_{y1}}{Re} \\ &\quad -(V_c)_t-\alpha^2\frac{V_c}{Re}, \end{aligned} \tag{47}$$

where we use (36) for expressing V_{yy} at the top wall in terms of $u_{y1}=u_y(k_x,1,k_z)$ and $W_{y1}=W_y(k_x,1,k_z)$ and the controller V_c.

Equation (45) can be solved in terms of integrals of the state and the boundary terms appearing in (46) and (47).

$$\begin{aligned} p &= -\frac{4\pi k_x i}{\alpha}\int_0^y U_y^e(\eta)\sinh\left(\alpha(y-\eta)\right)V(k_x,\eta,k_z)d\eta \\ &\quad +N\int_0^y\frac{\sinh\left(\alpha(y-\eta)\right)}{\alpha}V_y(k_x,\eta,k_z)d\eta \\ &\quad +2\pi i\frac{\cosh\left(\alpha(1-y)\right)}{\alpha\sinh\alpha}\frac{k_x u_{y0}+k_z W_{y0}}{Re} \\ &\quad +\frac{4\pi k_x i\cosh(\alpha y)}{\alpha\sinh\alpha}\int_0^1 U_y^e(\eta)\cosh\left(\alpha(1-\eta)\right) \\ &\quad \times V(k_x,\eta,k_z)d\eta-N\frac{\cosh(\alpha y)}{\alpha\sinh\alpha}\int_0^1\cosh\left(\alpha(1-\eta)\right) \\ &\quad \times V_y(k_x,\eta,k_z)d\eta-2\pi i\frac{\cosh(\alpha y)}{\alpha\sinh\alpha}\frac{k_x u_{y1}+k_z W_{y1}}{Re} \\ &\quad -\frac{\cosh(\alpha y)}{\alpha\sinh\alpha}\left((V_c)_t+\alpha^2\frac{V_c}{Re}\right). \end{aligned} \tag{48}$$

We proceed as in [26] and [10] and use the controller V_c, which appears *inside* the pressure solution (48), to make the pressure strict-feedback (spatially causal in y), which is a necessary structure for the application of a backstepping boundary controller [21]. Since the first three lines in (48) are already spatially causal, we need to cancel the fourth, fifth and sixth lines of (48). Set

$$\begin{aligned} (V_c)_t &= \alpha^2\frac{V_c}{Re}+2\pi i\frac{k_x(u_{y0}-u_{y1})+k_z(W_{y0}-W_{y1})}{Re} \\ &\quad +4\pi k_x i\int_0^1 U_y^e(\eta)\cosh\left(\alpha(1-\eta)\right)V(k_x,\eta,k_z)d\eta \\ &\quad -N\int_0^1\cosh\left(\alpha(1-\eta)\right)V_y(k_x,\eta,k_z)d\eta, \end{aligned} \tag{49}$$

which can be written as

$$\begin{aligned} (V_c)_t &= \alpha^2\frac{V_c}{Re}+2\pi i\frac{k_x(u_{y0}-u_{y1})+k_z(W_{y0}-W_{y1})}{Re} \\ &\quad -NV_c+\int_0^1\cosh\left(\alpha(1-\eta)\right)V(k_x,\eta,k_z) \\ &\quad \times\left(N+4\pi k_x i U_y^e(\eta)\right)d\eta. \end{aligned} \tag{50}$$

Then, the pressure is written in terms of a strict-feedback integral of the state V and the boundary terms u_{y0}, W_{y0} (proportional to the skin friction at the bottom) as follows

$$\begin{aligned} p &= -\frac{4\pi k_x i}{\alpha}\int_0^y U_y^e(\eta)\sinh\left(\alpha(y-\eta)\right)V(k_x,\eta,k_z)d\eta \\ &\quad -2\pi i\frac{\cosh(\alpha y)-\cosh\left(\alpha(1-y)\right)}{Re\alpha\sinh\alpha}\left(k_x u_{y0}+k_z W_{y0}\right) \\ &\quad +N\int_0^y\frac{\sinh\left(\alpha(y-\eta)\right)}{\alpha}V_y(k_x,\eta,k_z)d\eta. \end{aligned} \tag{51}$$

Similarly, solving for ϕ in terms of the control Φ_c and the right hand side of its Poisson equation (37),

$$\begin{aligned} \phi &= \frac{2\pi i}{\alpha}\int_0^y\sinh\left(\alpha(y-\eta)\right)\left(k_z u(k_x,\eta,k_z)\right. \\ &\quad \left.-k_x W(k_x,\eta,k_z)\right)d\eta+\frac{\sinh(\alpha y)}{\sinh\alpha}\Phi_c(k_x,k_y) \\ &\quad -\frac{2\pi i\sinh(\alpha y)}{\alpha\sinh\alpha}\int_0^1\sinh\left(\alpha(1-\eta)\right)\left(k_z u(k_x,\eta,k_z)\right. \\ &\quad \left.-k_x W(k_x,\eta,k_z)\right)d\eta. \end{aligned} \tag{52}$$

As in the pressure, an actuator (Φ_c in this case) appears inside the solution for the potential. The last two lines of (52) are non-strict-feedback integrals and need to be cancelled to apply the backstepping method. For this we use Φ_c by setting

$$\begin{aligned} \Phi_c(k_x,k_y) &= \frac{2\pi i}{\alpha}\int_0^1\sinh\left(\alpha(1-\eta)\right)\left(k_z u(k_x,\eta,k_z)\right. \\ &\quad \left.-k_x W(k_x,\eta,k_z)\right)d\eta. \end{aligned} \tag{53}$$

Then the potential can be expressed as a strict-feedback integral of the states u and W as follows

$$\begin{aligned} \phi &= \frac{2\pi i}{\alpha}\int_0^y\sinh\left(\alpha(y-\eta)\right)\left(k_z u(k_x,\eta,k_z)\right. \\ &\quad \left.-k_x W(k_x,\eta,k_z)\right)d\eta. \end{aligned} \tag{54}$$

Introducing the expressions (51) and (54) in (33) and (35), we get

$$\begin{aligned} u_t &= \frac{-\alpha^2 u+u_{yy}}{Re}-\beta(y)u-U_y^e(y)V-Nu-4\pi^2 k_x \\ &\quad \times\frac{\cosh(\alpha y)-\cosh\left(\alpha(1-y)\right)}{Re\alpha\sinh\alpha}\left(k_x u_{y0}+k_z W_{y0}\right) \\ &\quad -\frac{8\pi k_x^2}{\alpha}\int_0^y U_y^e(\eta)\sinh\left(\alpha(y-\eta)\right)V(k_x,\eta,k_z)d\eta \\ &\quad -2\pi i k_x N\int_0^y\frac{\sinh\left(\alpha(y-\eta)\right)}{\alpha}V_y(k_x,\eta,k_z)d\eta \\ &\quad -\frac{4\pi^2 k_z N}{\alpha}\int_0^y\sinh\left(\alpha(y-\eta)\right) \\ &\quad \times\left(k_z U(k_x,\eta,k_z)-k_x W(k_x,\eta,k_z)\right)d\eta, \end{aligned} \tag{55}$$

$$W_t = \frac{-\alpha^2 W+W_{yy}}{Re}-\beta(y)W-NW-4\pi^2 k_z$$

$$\times \frac{\cosh(\alpha y) - \cosh(\alpha(1-y))}{Re\alpha \sinh\alpha}(k_x u_{y0} + k_z W_{y0})$$
$$-\frac{8\pi k_x k_z}{\alpha}\int_0^y U_y^e(\eta)\sinh(\alpha(y-\eta))V(k_x,\eta,k_z)d\eta$$
$$-2\pi i k_z N\int_0^y \frac{\sinh(\alpha(y-\eta))}{\alpha}V_y(k_x,\eta,k_z)d\eta$$
$$+\frac{4\pi^2 k_x N}{\alpha}\int_0^y \sinh(\alpha(y-\eta))$$
$$\times (k_z U(k_x,\eta,k_z) - k_x W(k_x,\eta,k_z))\, d\eta. \quad (56)$$

We have omitted the equation for V since, from (36) and using the fact that $V(k_x,0,k_z)=0$, V is computed as

$$V = -2\pi i\int_0^y (k_x U(k_x,\eta,k_z) + k_z W(k_x,\eta,k_z))\, d\eta. \quad (57)$$

Now we use the following change of variables and its inverse,

$$Y = 2\pi i(k_x u + k_z W),\ \omega = 2\pi i(k_z u - k_x W), \quad (58)$$
$$u = \frac{2\pi i}{\alpha^2}(k_x Y + k_z\omega),\ W = \frac{2\pi i}{\alpha^2}(k_z Y - k_x\omega). \quad (59)$$

Defining $\epsilon = \frac{1}{Re}$ and the following functions

$$f = 4\pi i k_x\left\{\frac{U_y^e(y)}{2} + \int_\eta^y U_y^e(\sigma)\frac{\sinh(\alpha(y-\sigma))}{\alpha}d\sigma\right\}$$
$$+N\alpha\sinh(\alpha(y-\sigma)), \quad (60)$$
$$g = -\alpha\frac{\cosh(\alpha y) - \cosh(\alpha(1-y))}{Re\sinh\alpha}, \quad (61)$$
$$h_1 = 2\pi i k_z U_y^e, \quad (62)$$
$$h_2 = -N\alpha\sinh(\alpha(y-\eta)), \quad (63)$$

equations (55)–(56) expressed in terms of Y and ω are

$$Y_t = \epsilon\left(-\alpha^2 Y + Y_{yy}\right) - \beta(y)Y - NY + gY_{y0}$$
$$+\int_0^y f(k_x,y,\eta,k_z)Y(k_x,\eta,k_z)d\eta, \quad (64)$$
$$\omega_t = \epsilon\left(-\alpha^2\omega + \omega_{yy}\right) - \beta(y)\omega - N\omega$$
$$+h_1(y)\int_0^y Y(k_x,\eta,k_z)d\eta$$
$$+\int_0^y h_2(y,\eta)\omega(k_x,\eta,k_z)d\eta, \quad (65)$$

where we have used the inverse change of variables (59) to express u_{y0} and W_{y0} in terms of $Y_{y0} = Y_y(k_x,0,k_z)$ as follows

$$Y_{y0} = 2\pi i(k_x u_{y0} + k_z W_{y0}), \quad (66)$$

with boundary conditions

$$Y(t,k_x,0,k_z) = \omega(t,k_x,0,k_z) = 0, \quad (67)$$
$$Y(t,k_x,1,k_z) = Y_c(t,k_x,k_z) \quad (68)$$
$$\omega(t,k_x,1,k_z) = \omega_c(t,k_x,k_z), \quad (69)$$

where

$$Y_c = 2\pi i(k_x U_c + k_z W_c), \quad (70)$$
$$\omega_c = 2\pi i(k_z U_c - k_x W_c). \quad (71)$$

Equations (64)–(65) are a coupled, strict-feedback plant, with integral and reaction terms. As in [10], a variant of the design presented in [21] can be used to stabilize the system using a double backstepping transformation. The transformation maps, for each k_x and k_z, the variables (Y,ω) into the variables (Ψ,Ω), that verify the following family of heat equations (parameterized in k_x, k_z)

$$\Psi_t = \epsilon\left(-\alpha^2\Psi + \Psi_{yy}\right) - \beta(y)\Psi - N\psi, \quad (72)$$
$$\Omega_t = \epsilon\left(-\alpha^2\Omega + \Omega_{yy}\right) - \beta(y)\Omega - N\Omega, \quad (73)$$

with boundary conditions

$$\Psi(k_x,0,k_z) = \Psi(k_x,1,k_z) = 0, \quad (74)$$
$$\Omega(k_x,0,k_z) = \Omega(k_x,1,k_z) = 0. \quad (75)$$

The transformation is defined as follows,

$$\Psi = Y - \int_0^y K(k_x,y,\eta,k_z)Y(k_x,\eta,k_z)d\eta, \quad (76)$$
$$\Omega = \omega - \int_0^y \Gamma_1(k_x,y,\eta,k_z)Y(k_x,\eta,k_z)d\eta$$
$$-\int_0^y \Gamma_2(k_x,y,\eta,k_z)\omega(k_x,\eta,k_z)d\eta. \quad (77)$$

Following [10], [21], [26], the functions $K(k_x,y,\eta,k_z)$, $\Gamma_1(k_x,y,\eta,k_z)$, and $\Gamma_2(k_x,y,\eta,k_z)$ are found as the solution of the following partial integro-differential equations,

$$\epsilon K_{yy} = \epsilon K_{\eta\eta} + (\beta(y) - \beta(\eta))K - f$$
$$+\int_\eta^y f(\eta,\xi)K(y,\xi)d\xi, \quad (78)$$
$$\epsilon\Gamma_{1yy} = \epsilon\Gamma_{1\eta\eta} + (\beta(y) - \beta(\eta))\Gamma_1 - h_1 + \int_\eta^y \Gamma_2(y,\xi)$$
$$\times h_1(\xi)d\xi + \int_\eta^y f(\eta,\xi)\Gamma_1(y,\xi)d\xi, \quad (79)$$
$$\epsilon\Gamma_{2yy} = \epsilon\Gamma_{2\eta\eta} + (\beta(y) - \beta(\eta))\Gamma_2 - h_2$$
$$+\int_\eta^y h_2(\xi,\eta)\Gamma_2(y,\xi)d\xi. \quad (80)$$

Equations (78)–(80) are hyperbolic partial integro-differential equation in the region $\mathcal{T} = \{(y,\eta) : 0 \le y \le 1, 0 \le \eta \le y\}$. Their boundary conditions are

$$K(y,y) = -\frac{g(0)}{\epsilon}, \quad (81)$$
$$K(y,0) = \frac{\int_0^y K(y,\eta)g(\eta)d\eta - g(y)}{\epsilon}, \quad (82)$$
$$\Gamma_1(y,y) = 0, \quad (83)$$
$$\Gamma_1(y,0) = \frac{\int_0^y \Gamma_1(y,\eta)g(\eta)d\eta}{\epsilon}, \quad (84)$$
$$\Gamma_2(y,y) = 0,\ \Gamma_2(y,0) = 0. \quad (85)$$

Remark 1: Equations (78)–(85) are well-posed and can be solved symbolically, by means of a successive approximation series, or numerically [10], [21]. Note that (78) and (80) are autonomous. Hence, one must solve first for $K(k_x,y,\eta,k_z)$ and $\Gamma_2(k_x,y,\eta,k_z)$. Then the solution for Γ_2 is plugged in Equation 79 which then can be solved for $\Gamma_1(k_x,y,\eta,k_z)$.

Control laws Y_c and W_c are found evaluating (76)–(77) at $y=1$ and using (68)–(69) and (74)–(75), which yields

$$Y_c(t,k_x,k_z) = \int_0^1 K(k_x,1,\eta,k_z)Y(k_x,\eta,k_z)d\eta, \quad (86)$$

$$\omega_c(t,k_x,k_z) = \int_0^1 \Gamma_1(k_x,1,\eta,k_z)Y(k_x,\eta,k_z)d\eta + \int_0^1 \Gamma_2(k_x,1,\eta,k_z)\omega(k_x,\eta,k_z)d\eta. \quad (87)$$

Using (58)–(59) to write (86)–(87) in (u,W), we get

$$U_c = \int_0^1 K^{Uu}(k_x,1,\eta,k_z)u(k_x,\eta,k_z)d\eta + \int_0^1 K^{UW}(k_x,1,\eta,k_z)W(k_x,\eta,k_z)d\eta, \quad (88)$$

$$W_c = \int_0^1 K^{Wu}(k_x,1,\eta,k_z)u(k_x,\eta,k_z)d\eta + \int_0^1 K^{WW}(k_x,1,\eta,k_z)W(k_x,\eta,k_z)d\eta, \quad (89)$$

where

$$\begin{pmatrix} K^{Uu} \\ K^{UW} \\ K^{Wu} \\ K^{WW} \end{pmatrix} = \mathbf{A} \begin{pmatrix} K(k_x,y,\eta,k_z) \\ \Gamma_1(k_x,y,\eta,k_z) \\ 0 \\ \Gamma_2(k_x,y,\eta,k_z) \end{pmatrix}, \quad (90)$$

and where the matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = -\frac{4\pi^2}{\alpha^2} \begin{pmatrix} k_x^2 & k_xk_z & k_xk_z & k_z^2 \\ k_xk_z & k_z^2 & -k_x^2 & -k_xk_z \\ k_xk_z & -k_x^2 & k_z^2 & -k_xk_z \\ k_z^2 & -k_xk_z & -k_xk_z & k_x^2 \end{pmatrix}. \quad (91)$$

Stability in the controlled wave number range follows from stability of (72)–(73) and the invertibility of the transformation (76)–(77). We get the following result, whose proof we sketch (see [10] for more details).

Proposition 5.1: For $k_x^2+k_z^2 \le M^2$, the equilibrium $u \equiv V \equiv W \equiv 0$ of system (33)–(42) with control laws (50), (53), (88)–(89) is exponentially stable in the L^2 norm, i.e.,

$$\int_0^1 \left(|u|^2+|V|^2+|W|^2\right)(t,k_x,y,k_z)dy \le C_1 \mathrm{e}^{-2\epsilon t} \int_0^1 \left(|u|^2+|V|^2+|W|^2\right)(0,k_x,y,k_z)dy, \quad (92)$$

where $C_1 \ge 0$.

Proof: From equations (72)–(73) we get, using a standard Lyapunov argument,

$$\int_0^1 \left(|\Psi|^2+|\Omega|^2\right)(t,k_x,y,k_z)dy \le \mathrm{e}^{-2\epsilon t} \int_0^1 \left(|\Psi|^2+|\Omega|^2\right)(0,k_x,y,k_z)dy, \quad (93)$$

and then from the transformation (76)–(77) and its inverse (which is guaranteed to exist [21]), we get that

$$\int_0^1 \left(|Y|^2+|\omega|^2\right)(t,k_x,y,k_z)dy \le C_0 \mathrm{e}^{-2\epsilon t} \int_0^1 \left(|Y|^2+|\omega|^2\right)(0,k_x,y,k_z)dy, \quad (94)$$

where $C_0 > 0$ is a constant depending on the kernels K, Γ_1 and Γ_2 and their inverses. Then writing (u,W) in terms of (Y,ω) and bounding the norm of V by the norm of Y (using $Y=-V_y$ and Poincare's inequality), the result follows. ■

B. Uncontrolled wave number analysis

When $k_x^2+k_z^2 > M$, plant verifies the following equations

$$u_t = \frac{-\alpha^2 u + u_{yy}}{Re} - \beta(y)u - U_y^e(y)V - 2\pi k_x ip + 2\pi k_z iN\phi - Nu, \quad (95)$$

$$V_t = \frac{-\alpha^2 V + V_{yy}}{Re} - \beta(y)V - p_y, \quad (96)$$

$$W_t = \frac{-\alpha^2 W + W_{yy}}{Re} - \phi W - 2\pi k_z ip - 2\pi k_x iN\phi - NW, \quad (97)$$

the Poisson equation for the potential

$$-\alpha^2\phi + \phi_{yy} = 2\pi i\left(k_z u - k_x W\right) \quad (98)$$

the continuity equation

$$2\pi i k_x u + V_y + 2\pi k_z W = 0, \quad (99)$$

and Dirichlet boundary conditions

$$u(t,k_x,0,k_y) = V(t,k_x,0,k_y) = W(t,k_x,0,k_y) = 0, \quad (100)$$

$$u(t,k_x,1,k_y) = V(t,k_x,1,k_y) = W(t,k_x,1,k_y) = 0, \quad (101)$$

$$\phi(t,k_x,0,k_y) = \phi(t,k_x,1,k_y) = 0. \quad (102)$$

Using (58), one gets the following equations for Y and ω.

$$Y_t = \epsilon\left(-\alpha^2 Y + Y_{yy}\right) - \beta(y)Y - 2\pi k_x iU_y^e(y)V + \alpha^2 p - NY, \quad (103)$$

$$\omega_t = \epsilon\left(-\alpha^2\omega + \omega_{yy}\right) - \beta(y)\omega - 2\pi k_z iU_y^e(y)V - \alpha^2 N\phi - N\omega. \quad (104)$$

The Poisson equation for the potential is, in terms of ω,

$$-\alpha^2\phi + \phi_{yy} = \omega. \quad (105)$$

Consider the Lyapunov function

$$\Lambda = \int_0^1 \frac{|u|^2+|V|^2+|W|^2}{2}dy, \quad (106)$$

where we write $\int_0^1 f = \int_0^1 f(k_x,y,k_z)dy$. The function Λ is the L^2 norm (kinematic energy) of the velocity field.

Denote by f^* the complex conjugate of f. Substituting Y and ω from (59) into (106), we get

$$\Lambda = \int_0^1 4\pi^2 \left[\frac{k_x^2|Y|^2+k_z^2|\omega|^2+k_xk_z(Y^*\omega+Y\omega^*)}{2\alpha^4} + \frac{k_z^2|Y|^2+k_x^2|\omega|^2-k_xk_z(Y^*\omega+Y\omega^*)}{2\alpha^4}\right]dy + \int_0^1 \frac{|V|^2}{2}dy = \int_0^1 \frac{|Y|^2+|\omega|^2+\alpha^2|V|^2}{2\alpha^2}dy. \quad (107)$$

Define then a new Lyapunov function,

$$\Lambda_1 = \alpha^2\Lambda = \int_0^1 \frac{|Y|^2+|\omega|^2+\alpha^2|V|^2}{2}dy. \quad (108)$$

The time derivative of Λ_1 can be estimated as follows,

$$\begin{aligned}\dot{\Lambda}_1 &= -2\epsilon\alpha^2\Lambda_1 - \epsilon\int_0^1 \left(|Y_y|^2 + |\omega_y|^2 + \alpha^2|V_y|^2\right) \\ &\quad -N\int_0^1 \left(|Y|^2+|\omega|^2\right) - \alpha^2 N\int_0^1 \frac{\phi^*\omega+\phi\omega^*}{2} \\ &\quad +\int_0^1 \pi i U_y^e(y)V^*(2k_xY+k_z\omega) \\ &\quad -\int_0^1 \pi i U_y^e(y)V(2k_xY^*+k_z\omega^*) \\ &\quad +\alpha^2\int_0^1 \frac{P^*Y+PY^*-P_y^*V-P_yV^*}{2}. \qquad (109)\end{aligned}$$

For bounding (109), we use the following two lemmas.

Lemma 5.1:

$$-\alpha^2\int_0^1 \frac{\phi^*\omega+\phi\omega^*}{2} \le \int_0^1 |\omega|^2. \qquad (110)$$

Proof: The term we want to estimate is

$$-\alpha^2\int_0^1 \frac{\phi^*\omega+\phi\omega^*}{2}. \qquad (111)$$

Substituting $\alpha^2\phi$ from (105), (111) can be written as

$$-\int_0^1 \frac{\phi_{yy}^*\omega+\phi_{yy}\omega^*}{2} + \int_0^1 |\omega|^2. \qquad (112)$$

Therefore, we need to prove that

$$\int_0^1 \left(\phi_{yy}^*\omega+\phi_{yy}\omega^*\right) \ge 0. \qquad (113)$$

Substituting ω from equation (105) into (113), we get

$$\begin{aligned}&\int_0^1 \left(\phi_{yy}^*\omega+\phi_{yy}\omega^*\right) \\ &= \int_0^1 |\phi_{yy}|^2 - \alpha^2\int_0^1 \left(\phi_{yy}^*\phi+\phi_{yy}\phi^*\right) \\ &= \int_0^1 |\phi_{yy}|^2 + \alpha^2\int_0^1 |\phi_y|^2, \qquad (114)\end{aligned}$$

which is nonnegative. ■

Lemma 5.2:

$$|U_y^e(y)| \le 4+H. \qquad (115)$$

Proof: Computing $U_y^e(y)$ from (15),

$$U_y^e(y) = H\frac{\cosh(Hy)-\cosh(H(1-y))}{2\sinh H/2 - \sinh H}. \qquad (116)$$

Calling $g_1(y) = \cosh(Hy)-\cosh(H(1-y))$, since $g_1'(y) = H\left(\sinh(Hy)+\sinh(H(1-y))\right)$ is always positive for $y\in(0,1)$, the maximum must be in the boundaries. Therefore

$$|U_y^e(y)| \le g_2(H) = H\frac{\cosh H - 1}{\sinh H - 2\sinh H/2}. \qquad (117)$$

One can rewrite g_2 as

$$g_2 = H\frac{\sinh H/2}{\cosh H/2 - 1}. \qquad (118)$$

Since $g_2(0)=4$, it suffices to verify that $g_2'(H)\le 1$.

$$g_2'(H) = \frac{g_3}{g_4} = \frac{\sinh H/2 - H^2/2}{\cosh H/2 - 1}. \qquad (119)$$

This is equivalent to verify that $g_3 \le g_4$. Since $g_3(0) = g_4(0) = 0$, it is enough that $g_3' \le g_4'$, which follows from

$$g_3' = H/2\left(\cosh H/2 - 2H\right) \le H/2(\sinh H/2) = g_4', \qquad (120)$$

because $\cosh x - 4x \le \sinh x$. ■

Integrating by parts and applying Lemma 5.1,

$$\begin{aligned}\dot{\Lambda}_1 &\le -2\epsilon\alpha^2\Lambda_1 - \epsilon\int_0^1 \left(|Y_y|^2+|\omega_y|^2+\alpha^2|V_y|^2\right) \\ &\quad +\int_0^1 \pi i U_y^e(y)V^*(k_xY+k_z\omega) \\ &\quad -\int_0^1 \pi i U_y^e(y)V(k_xY^*+k_z\omega^*) - N\int_0^1 |Y|^2. \qquad (121)\end{aligned}$$

Using Lemma 5.2 to bound U_y^e in (121),

$$\begin{aligned}\dot{\Lambda}_1 &\le -2\epsilon\left(1+\alpha^2\right)\Lambda_1 - N\int_0^1 |Y|^2 dy \\ &\quad +2\pi\left(4+H\right)\int_0^1 \left(|V|(|k_x||Y|+|k_z||\omega|\right)dy \\ &\le \left(4+H-2\epsilon\left(1+\alpha^2\right)\right)\Lambda_1 \qquad (122)\end{aligned}$$

where we have applied Young's and Poincare's inequalities. Hence, if $\alpha^2 \ge \frac{4+H}{2\epsilon}$,

$$\dot{\Lambda}_1 \le -2\epsilon\Lambda_1. \qquad (123)$$

Dividing (123) by α^2 and using (108), we get that

$$\dot{\Lambda} \le -2\epsilon\Lambda, \qquad (124)$$

and stability in the uncontrolled wave number range follows when $k_x^2+k_z^2 \ge M^2$ for M (conservatively) chosen as

$$M \ge \frac{1}{2\pi}\sqrt{\frac{(H+4)Re}{2}}. \qquad (125)$$

We summarize the result in the following proposition.

Proposition 5.2: For $k_x^2+k_z^2 \ge M^2$ where $M \ge \frac{1}{2\pi}\sqrt{\frac{(H+4)Re}{2}}$, the equilibrium $u\equiv V\equiv W\equiv 0$ of the uncontrolled system (95)–(102) is exponentially stable in the L^2 sense, i.e.,

$$\begin{aligned}&\int_0^1 \left(|u|^2+|V|^2+|W|^2\right)(t,k_x,y,k_z)dy \\ &\le \mathrm{e}^{-2\epsilon t}\int_0^1 \left(|u|^2+|V|^2+|W|^2\right)(0,k_x,y,k_z)dy. \qquad (126)\end{aligned}$$

C. Main result

Substituting (50), (53) and (88)–(89) in (44), and using the Fourier convolution theorem, we get the control laws in physical space, which can be expressed compactly as

$$\begin{aligned}\begin{pmatrix} U_c \\ W_c \\ \Phi_c \end{pmatrix} &= \int_{-\infty}^{\infty}\int_0^1\int_{-\infty}^{\infty} \mathbf{\Sigma}(x-\xi,\eta,z-\zeta) \\ &\quad \times\begin{pmatrix} u(\xi,\eta,\zeta) \\ W(\xi,\eta,\zeta)\end{pmatrix} d\xi d\eta d\zeta, \qquad (127)\end{aligned}$$

where

$$\begin{aligned}\mathbf{\Sigma}(\xi,\eta,\zeta) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{\Sigma}(k_x,\eta,k_z) \\ &\quad \times\chi(k_x,k_z)\mathrm{e}^{2\pi i(k_x\xi+k_z\zeta)}dk_zdk_x, \qquad (128)\end{aligned}$$

and

$$\mathbf{\Sigma}=\begin{pmatrix} K^{Uu}(k_x,1,\eta,k_z) & K^{UW}(k_x,1,\eta,k_z) \\ K^{Wu}(k_x,1,\eta,k_z) & K^{WW}(k_x,1,\eta,k_z) \\ \frac{2\pi i k_z \sinh(\alpha(1-\eta))}{\alpha} & -\frac{2\pi i k_k \sinh(\alpha(1-\eta))}{\alpha} \end{pmatrix}. \quad (129)$$

Control law V_c is a dynamic feedback law computed as the solution of the following forced parabolic equation

$$(V_c)_t = \frac{(V_c)_{xx}+(V_c)_{zz}}{Re} - NV_c + g(t,x,z), \quad (130)$$

where $g(t,x,z)$ is defined as

$$g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Bigg[\int_0^1 g_V(x-\xi,\eta,z-\zeta)V(\xi,\eta,\zeta)d\eta \\ +g_W(x-\xi,z-\zeta)\left(W_y(\xi,0,\zeta)-W_y(\xi,1,\zeta)\right) \\ +g_u(x-\xi,z-\zeta)\left(u_y(\xi,0,\zeta)-u_y(\xi,1,\zeta)\right)\Bigg]d\xi d\zeta, (131)$$

and

$$g_u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi i\frac{k_x}{Re}\chi(k_x,k_z)\mathrm{e}^{2\pi i(k_x\xi+k_z\zeta)}dk_z dk_x, (132)$$

$$g_V = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cosh\left(\alpha(1-\eta)\right)\left(N+4\pi k_x i U_y^e(\eta)\right) \\ \times\chi(k_x,k_z)\mathrm{e}^{2\pi i(k_x\xi+k_z\zeta)}dk_z dk_x, \quad (133)$$

$$g_W = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2\pi i\frac{k_z}{Re}\chi(k_x,k_z)\mathrm{e}^{2\pi i(k_x\xi+k_z\zeta)}dk_z dk_x. (134)$$

As in [10], [26], considering all wave numbers and using Proposition 5.1 and Proposition 5.2, the following result holds regarding the convergence of the closed-loop system.

Theorem 1: Consider the system (21)–(42) with control laws (127)–(134). Then the equilibrium profile $u \equiv V \equiv W \equiv 0$ is asymptotically stable in the L^2 norm, i.e.,

$$\int_{-\infty}^{\infty}\int_0^1\int_{-\infty}^{\infty}\left(u^2+V^2+W^2\right)(t,x,y,z)dxdydz \\ \leq C_2\mathrm{e}^{-2\epsilon t} \\ \times\int_{-\infty}^{\infty}\int_0^1\int_{-\infty}^{\infty}\left(u^2+V^2+W^2\right)(0,x,y,z)dxdydz. (135)$$

where $C_2 = \max\{C_1, 1\} \geq 0$.

We have assumed in the above result that the closed loop linearized system is well-posed and that the velocity and electromagnetic field equations have at least L^2 solutions. See [20] for some mathematical considerations on the well-posedness of MHD problems.

Control laws (127)–(134) of Theorem 1 require full-state knowledge. In [27] we presented an observer for estimation of velocity and electromagnetic fields of the Hartmann flow, based on boundary measurement of pressure, current and skin friction. Such an observer can be used together with the control laws (127)–(134) to obtain an output feedback stabilizing boundary controller that only needs boundary measurements.

REFERENCES

[1] O. M. Aamo and M. Krstic, *Flow Control by Feedback: Stabilization and Mixing*, Berlin:Springer, 2002.

[2] J. Baker, A. Armaou and P.D. Christofides, "Nonlinear control of incompressible fluid flow: application to Burgers' equation and 2D channel flow," *J. Math. Anal. Appl.*, vol. 252, pp. 230–255, 2000.

[3] A. Balogh, W.-J. Liu and M. Krstic M, "Stability enhancement by boundary control in 2D channel flow," *IEEE Transactions on Automatic Control*, vol. 46, pp. 1696–1711, 2001.

[4] J. Baker, A. Armaou and P.D. Christofides, "Drag reduction in incompressible channel flow using electromagnetic forcing," *Proceedings of the 2000 ACC*, Chicago, pp. 4269–4273, 2000.

[5] B. Bamieh, F. Paganini and M.A. Dahleh M. A. , "Distributed control of spatially-invariant systems," *IEEE Trans. Automatic Control*, vol 45, pp. 1091–1107, 2000.

[6] V. Barbu, C. Popa, T. Havarneanu and S.S. Sritharan, "Exact controllability of magneto-hydrodynamic equations," *Communications on Pure and Applied Mathematics*, vol. 56(6), pp. 732–783, 2003

[7] V. Barbu, "Stabilization of a plane channel flow by wall normal controllers" Preprint, 2006.

[8] T.W. Berger, J. Kim, C. Lee and J. Lim, "Turbulent boundary layer control utilizing the Lorentz force," *Phys. Fluids*, vol. 12, p. 631, 2000.

[9] K.S. Breuer and J. Park, "Actuation and control of a turbulent channel flow using Lorentz forces," *Phys. Fluids*, vol. 16(4), p. 897, 2004.

[10] J. Cochran, R. Vazquez and M. Krstic, "Backstepping boundary control of Navier-Stokes channel flow: a 3D extension," *Proceedings of the 25th American Control Conference (ACC)*, Minneapolis, 2006.

[11] K. Debbagh, P. Cathalifaud and C. Airiau, "Optimal and robust control of small disturbances in a channel flow with a normal magnetic field," preprint, 2006.

[12] M. Hogberg, T.R. Bewley and D.S. Henningson, "Linear feedback control and estimation of transition in plane channel flow," *Journal of Fluid Mechanics*, vol. 481, pp.149–175, 2003.

[13] M.R. Jovanovic and B. Bamieh, "Componentwise energy amplification in channel flows," *J. Fluid Mech.*, vol. 543, pp. 145–183, 2005.

[14] M. Krstic, I. Kanellakopoulos and P.V. Kokotovic, *Nonlinear and Adaptive Control Design*, New York:Wiley, 1995.

[15] D. Lee and H. Choi, "Magnetohydrodynamic turbulent flow in a channel at low magnetic Reynolds number," *Journal of Fluid Mechanics*, vol. 439, pp. 367–394, 2001.

[16] U. Muller and L. Buhler, *Magnetofluiddynamics in Channels and Containers*, Berlin:Springer, 2001.

[17] J. Pang and K.-S. Choi, "Turbulent drag reduction by Lorentz force oscillation," *Phys. Fluids*, vol. 16(5), L35, 2004.

[18] E. Schuster and M. Kristic, "Inverse optimal boundary control for mixing in magnetohydrodynamic channel flows," *Proceedings of the 2003 CDC*, Maui, Hawaii USA, 2003.

[19] P.J. Schmid and D.S. Henningson, *Stability and Transition in Shear Flows*, Berlin:Springer, 2001.

[20] M. Sermange and R. Temam, "Some mathematical questions related to the MHD equations," *Communications on Pure and Applied Mathematics*, vol. 36, pp. 635–664, 1983.

[21] A. Smyshlyaev and M. Krstic, "Closed form boundary state feedbacks for a class of partial integro-differential equations," *IEEE Transactions on Automatic Control*, vol. 49, pp. 2185–2202, 2004.

[22] S.S. Sritharan, V. Barbu, T. Havarneanu and C. Popa, "Local controllability for the magnetohydrodynamic equations revisited," *Advances in Differential Equations*, vol. 10(5), pp. 481–504, 2005.

[23] J.-P. Thibault and L. Rossi, "Electromagnetic flow control: characteristic numbers and flow regimes of a wall-normal actuator," *J. Phys. D: Appl. Phys.*, vol. 36, pp. 2559–2568, 2003.

[24] R. Triggiani, "Exponential feedback stabilization of a 2-D linearized Navier-Stokes channel flow by finite-dimensional, wall-normal boundary controllers, with arbitrarily small support," preprint, 2006

[25] R. Vazquez, E. Trélat and J.-M. Coron, "Stable Poiseuille flow transfer for a Navier-Stokes system," *Proc. of the 2006 ACC*, Minneapolis, 2006.

[26] R. Vazquez and M. Krstic, "A closed-form feedback controller for stabilization of linearized Navier-Stokes equations: the 2D Poiseuille flow," *Proceedings of the 2005 CDC*, Sevilla, Spain, 2005.

[27] R. Vazquez and M. Krstic, "A closed-form observer for the 3D inductionless MHD and Navier-Stokes channel flow," *Proceedings of the 2006 CDC*, San Diego, 2006.