Inverse Optimal Boundary Control for Mixing in Magnetohydrodynamic Channel Flows*

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***Abstract*— A nonlinear Lyapunov-based boundary control law is proposed for mixing enhancement in a magnetohydrodynamic (MHD) channel flow, also known as Hartmann flow. This flow is characterized by an electrically conducting fluid moving between parallel plates in presence of an externally imposed transverse magnetic field. The system is described by the MHD equations, a combination of the Navier-Stokes equation and the Maxwell equations under the so-called MHD approximation. Micro-jets and pressure and magnetic field sensors embedded into the walls of the flow domain are considered in this work to find a feedback control law that is optimal with respect to a cost functional related to mixing measure.**

I. INTRODUCTION

Recent years have been marked by dramatic advances in active flow control (see [1] and the references therein) and their impact in aircraft aerodynamics and propulsion. Flow control, implemented through MEMS actuactors and sensors, and employing modern nonlinear and adaptive control algorithms, can be effective in reducing drag and separation over aircraft wings, eliminating instabilities in various sections of jet engines (inlet, compressor rotating stall), supressing instabilities within jet engine combustion chambers (thermoacoustic oscillations, flamefront instabilities, etc.), reducing jet noise, reducing thermal signature of jet exhaust through actively controlled mixing, and for steering of the overall vehicle. Equally exciting applications can be quoted from industrial process control, petroleum engineering, and even bio-medical engineering.

Up until now flow control developments have had little impact on electrically conducting fluids moving in electromagnetic fields. Employing control can qualitatively change the dynamic behavior of this kind of flows in fusion, hypersonic flight, propulsion and laser applications. Generally speaking, active control can be used to suppress turbulence where undesirable or enhance it when mixing is desired. As a result, a small amount of active control can greatly influence the heat transfer characteristics of a system (fusion, propulsion) or the external power needed to sustain its operation stably (hypersonic flight, lasers, propulsion, fusion).

The possible usage of liquid metals or electrically conducting liquid salts as self-cooled blankets in magnetic confinement fusion reactors has been in consideration for the past 30 years. The main function of the coolant is the absorbtion of energy from the neutron flux and the transfer of heat to an external energy conversion system. In addition, if a breeder liquid metal such as liquid-lithium is considered, the blanket can also carry out the breeding of tritium, which is part of the fuel used by the reactor.

The liquid metal flow is affected by the strong magnetic field (5-10 Tesla) used to confine the plasma inside the reactor. The interaction between the flow and the strong imposed magnetic field generates very intense magnetohydrodynamic (MHD) effects. Among the most important effects, we find the increase of pressure drop and the decrease of heat transfer rate. When an electrically conducting fluid moves in the presence of a transverse magnetic field, it produces and electrical field due to charge separation and subsequently an electric current. The interaction between this created electric current and the imposed magnetic field originates a body force, called the Lorentz force, which acts on the fluid itself. Since this force acts in the opposite direction of the fluid motion, it is necessary to increase the pressure drop to mantain the mean velocity of the flow and a high increase of power is required to pump the liquid through the ducts forming the blanket. In addition, this force tends to supress turbulence and laminarize the flow, reducing the heat transfer rate as consequence. A good review of the present state of research in the field can be found in [2]. It is possible to note that although extensive experimental and numerical work is going through, the use of feedback to increase the heat transfer rate without increasing the pressure drop has not yet been explored.

The application of MHD flow control in aerospace engineering was already considered in the mid-1950's. This was coincident to the first studies on the problem of an aerospace vehicle reentering the atmosphere from space. The high temperature reached at the surface of the vehicle flying at hypersonic speed causes the ionization of the sorrounding air molecules and the consequent formation of a plasma. It was natural then to try to explote the plasma capability of interacting with an electromagnetic field. By imposing a suitable magnetic field, it is possible to modify the aerodynamic forces and heat transfer rates in a convenient way. Since the Lorentz force tends to oppose fluid motion across magnetic field lines, a transverse magnetic field applied to the plasma layer would tend to increase the drag braking the vehicle in atmospheric entry and reduce heat transfer and skin friction

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by slowing and laminarizing the flow near the surface of the body.

The use of this MHD principle would have been an alternative to the use of heat shield, the conventional approach to thermal protection for the last 40 years. But these ideas could not be put in practice because of the large, heavy magnets required to provide a magnetic field strong enough to affect the thermally ionized reentry flow characterized by a relatively low electrical conductivity. This subject has been revived by the appearance of superconducting magnets. It seems that the light-weight magnets, together with the utilization of artificial ionization, would give some new consideration to the usage of electromagnetic control techniques. Present work studies numerically the influence of the imposed magnetic field in different flows [3], [4], [5], [6], [7]. However, the feedback of some information of the system, such as variations in the current density, induced magnetic field or pressure, has not been considered to modulate the intensity of the imposed magnetic field. A serious analysis of the potential replacement of the strong magnetic field (large magnets) by a much weaker one (small magnets) conveniently modulated by feedback is still pending.

This paper is our first attempt at this exciting new research field. We consider a Hartmann flow, a electrically conducting fluid moving between parallel plates through an imposed transverse magnetic field. The fluid in this case is considered incompressible and Newtonian (constant viscosity). We use feedback boundary control to improve mixing by enhancing the instability of the Hartmann flow profile. From the point of view of sensors and actuators we follow the ideas introduced in previous work [8], [9], [10], [11]. Micro-jets and pressure and magnetic field sensors embedded into the walls of the flow domain are used to implement the feedback control law.

The paper is organized as follow. Section II introduces the governing equations of our system. The fully developed solution is presented in Section III and the perturbation equations are introduced in Section IV. The core of the paper is Section V, where the Lyapunov analysis for the designed boundary control law is presented. Section VI closes the paper stating the conclusion and the identified future work.

II. Governing Equations

Let us consider the flow of an incompressible conducting fluid between parallel planes where a magnetic field $\mathbf{B_o} = B_o\hat{\mathbf{y}}$ perpendicular to the channel axis is externally applied. In addition let us assume the presence of a uniform pressure gradient in the $-\hat{\mathbf{x}}$ direction. Figure 1 illustrates the configuration. This flow was first investigated experimentally and theoretically by Hartmann [12].

The governing equations for the stated problem are the transport equation of linear momentum

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla P + \rho\nu\nabla^2\mathbf{v} + \mathbf{f} + \mathbf{j}\times\mathbf{B}, \quad (1)$$

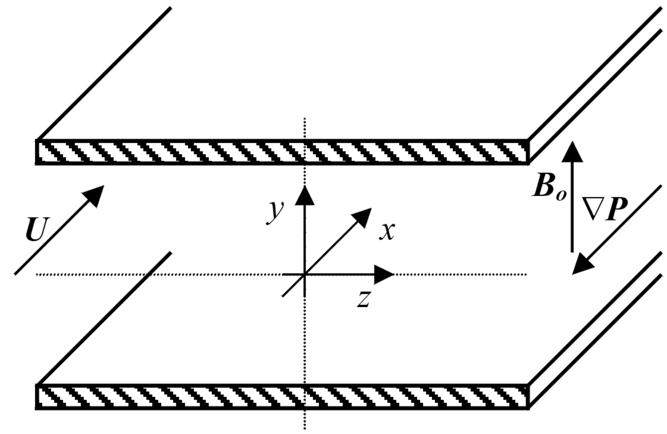

Fig. 1. Flow between parallel planes in presence of a transverse magnetic field (Hartmann flow).

and the transport equation of magnetic induction

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} = \frac{1}{\mu\sigma}\nabla^2\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{v}. \quad (2)$$

The flow velocity is denoted by $\mathbf{v}$, the magnetic field by $\mathbf{B}$ and the current density by $\mathbf{j}$ while P denotes the pressure, ρ the fluid mass density, ν the kinematic viscosity, μ the magnetic permeability and σ the electrical conductivity. The volumetric forces of non-electromagnetic origin is represented by $\mathbf{f}$ and the $\mathbf{j}\times\mathbf{B}$ term represents the Lorentz forces. The Lorentz forces couple the mechanical and electrodynamic states of the system and act in planes perpendicular to both current density and magnetic field vectors. Coulomb forces $q\mathbf{E}$, where q is the electrical charge and $\mathbf{E}$ the electrical field, are negligible in comparison to the Lorentz forces.

The magnetic induction equation is derived from Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \quad (3)$$

Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \quad (4)$$

Ampere's law

$$\mu\mathbf{j} = \nabla\times\mathbf{B}, \quad (5)$$

and the fact that $\mathbf{B}$ and $\mathbf{v}$ are solenoidal

$$\nabla\cdot\mathbf{B} = 0, \quad (6)$$

$$\nabla\cdot\mathbf{v} = 0. \quad (7)$$

In this work we consider the 2-D Hartmann flow. Figure 2 shows the geometrical arrangement. In this case we can write $\mathbf{v} = \mathbf{v}(x,y,t) = U(x,y,t)\hat{\mathbf{x}} + V(x,y,t)\hat{\mathbf{y}}$, $\mathbf{B} = \mathbf{B}(x,y,t) = B^u(x,y,t)\hat{\mathbf{x}} + B^v(x,y,t)\hat{\mathbf{y}}$ and $P = P(x,y,t)$.

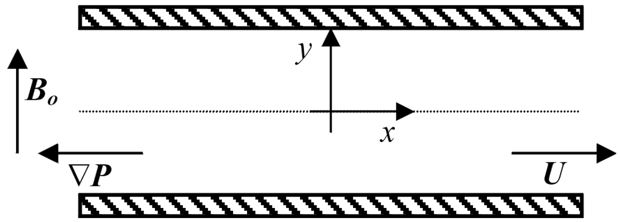

Fig. 2. 2-D Hartmann flow.

III. Equilibrium Solution

For channels with constant cross section, as the one depicted in Figure 1, a fully developed equilibrium flow is established. In this case, the flow velocity $\mathbf{v} = \bar{U}(y)\hat{\mathbf{x}}$ has only one component depending on the coordinate y. The magnetic field is decomposed into two contributions, one due to the external imposed magnetic field and the other caused by the magnetic field induced by the flow $\mathbf{B} = \mathbf{B_o} + \bar{\mathbf{b}} = B_o\hat{\mathbf{y}} + \bar{\mathbf{b}}$. Substituting this expression for the magnetic field $\mathbf{B}$ into equation (2) shows that the only component of the induced magnetic field is $\bar{\mathbf{b}} = \bar{b}(y)\hat{\mathbf{x}}$. The induction equation reduces then to

$$0 = \mu\sigma B_o \frac{d\bar{U}}{dy} + \frac{d^2\bar{b}}{dy^2}. \tag{8}$$

Using Ampere's law (5) it is possible to write the current density $\mathbf{j}$, and consequently the Lorentz force $\mathbf{j}\times\mathbf{B}$, in terms of $\bar{b}$. Then the momentum equation can be written as

$$0 = -\frac{d\bar{P}}{dx} + \frac{B_o}{\mu}\frac{d\bar{b}}{dy} + \rho\nu\frac{d^2\bar{U}}{dy^2}. \tag{9}$$

We consider viscous fluids with no slip at the fluid-wall interface Γ. Therefore the hydrodynamic boundary condition is

$$\mathbf{v} = 0 \quad \text{at} \;\; \Gamma, \tag{10}$$

which means that all the velocity components vanish at the wall. For walls with finite electrical conductivity σ_w, magnetic permeability μ_w and normal $\mathbf{n}$, the condition that the tangential component of the electrical field is continuos across the wall interface can be expressed in terms of $\bar{b}$ as [13]

$$\frac{\partial\bar{b}}{\partial n} - \frac{1}{c}\bar{b} = 0 \quad \text{at}\,\Gamma, \tag{11}$$

with the wall conductance ratio defined as $c = \frac{\mu_w\sigma_w t_w}{\mu\sigma L}$ where the wall thickness t_w is often small compared to the dimension of the cross section L. Two limiting cases can be considered

$$\begin{array}{lll} \bar{b} = 0 & \text{at}\,\Gamma\,\text{as}\,c\to 0 & \text{(perfectly insulating walls)} \\ \frac{\partial\bar{b}}{\partial n} = 0 & \text{at}\,\Gamma\,\text{as}\,c\to\infty & \text{(perfectly conducting walls).} \end{array}$$

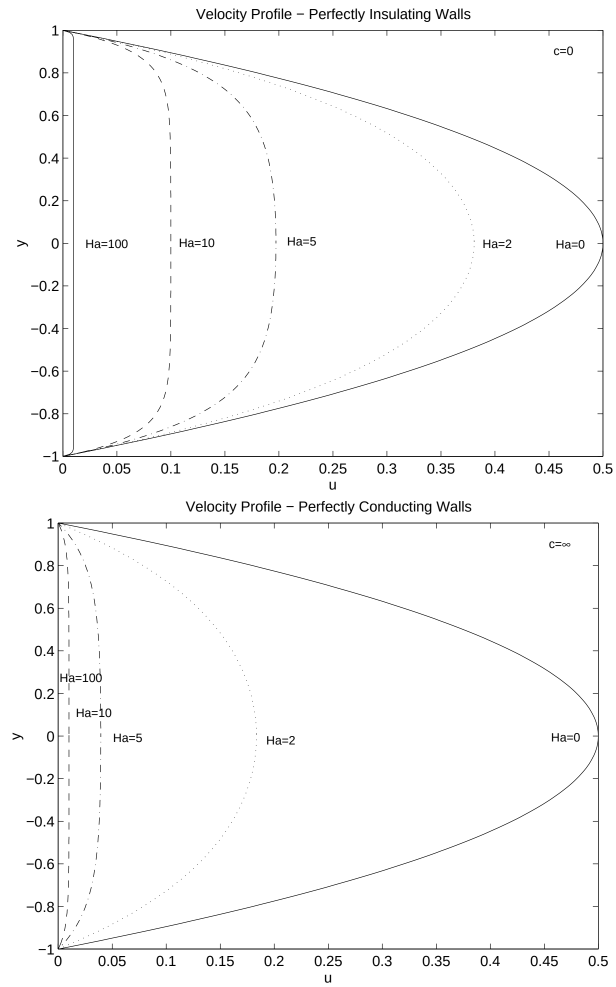

Fig. 3. Velocity profiles for Hartmann flow at Hartmann numbers $Ha = 0, 2, 5, 10, 100$ for perfectly insulating walls ($c = 0$) and for perfectly conducting walls ($c = \infty$).

Defining the dimensionless variables

$$y^* = \frac{y}{L} = \frac{y}{y_o} \tag{12}$$

$$\bar{U}^* = \frac{\bar{U}}{\frac{L^2}{\rho\nu}(-\frac{\partial\bar{P}}{\partial x})} = \frac{\bar{U}}{U_o} \tag{13}$$

$$\bar{b}^* = \frac{\bar{b}}{\mu L^2\sqrt{\frac{\sigma}{\rho\nu}}(-\frac{\partial\bar{P}}{\partial x})} = \frac{\bar{b}}{b_o} \tag{14}$$

we can rewrite equations (8) and (9) as

$$Ha\frac{d\bar{U}^*}{dy^*} + \frac{d^2\bar{b}^*}{dy^{*2}} = 0 \tag{15}$$

$$Ha\frac{d\bar{b}^*}{dy^*} + \frac{d^2\bar{U}^*}{dy^{*2}} = -1 \tag{16}$$

with boundary conditions (10) and (11) now expressed as

$$\begin{array}{lcll} \bar{U}^* & = & 0 & \text{at}\,y^* = \pm 1 \\ \mp\frac{d\bar{b}^*}{dy^*} - \frac{\bar{b}^*}{c} & = & 0 & \text{at}\,y^* = \pm 1 \end{array} \tag{17}$$

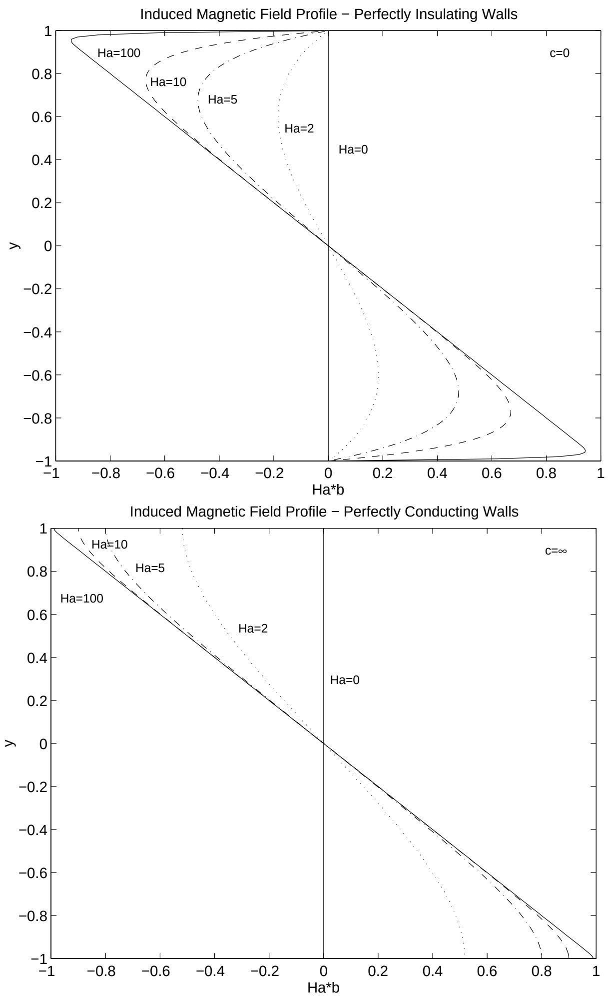

Fig. 4. Induced magnetic field profiles for Hartmann flow at Hartmann numbers $Ha = 0, 2, 5, 10, 100$ for perfectly insulating walls ($c = 0$) and for perfectly conducting walls ($c = \infty$).

where $Ha = B_o L \sqrt{\frac{\sigma}{\rho\nu}}$ is the Hartmann number. The solution for system (15) – (16) with boundary conditions (17) is given by

$$\begin{aligned} \bar{U}^*(y^*) &= \frac{1}{Ha}\frac{c+1}{cHa + \tanh(Ha)}\left[1 - \frac{\cosh(Ha\, y^*)}{\cosh(Ha)}\right] \\ \bar{b}^*(y^*) &= -\frac{y^*}{Ha} + \frac{1}{Ha}\frac{c+1}{cHa+\tanh(Ha)}\frac{\sinh(Ha\, y^*)}{\cosh(Ha)}. \end{aligned}$$

Figures 3 and 4 show respectively the velocity and induced magnetic field profiles for different values of the Hartmann number Ha and for perfectly insulating walls, $c = 0$, and perfectly conducting walls, $c = \infty$.

IV. Perturbation Equations

Dropping the star notation we can write the dimensionless momentum and induction equations as

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\nabla P + \frac{1}{R}\nabla^2\mathbf{v} + \mathbf{f} \\ &\quad + \frac{N}{R_m}\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right] \qquad (18) \end{aligned}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} = \frac{1}{R_m}\nabla^2\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{v}. \qquad (19)$$

Defining de deviation variables as

$$\begin{aligned} u &= U - \bar{U} \\ v &= V - \bar{V} = V \\ b^u &= B^u - \bar{B}^u = B^u - \bar{b} \\ b^v &= B^v - \bar{B}^v = B^v - B_o \\ p &= P - \bar{P} \end{aligned}$$

we can write the dimensionless perturbation equations as

$$\begin{aligned} &\frac{\partial u}{\partial t} + (\bar{U}+u)\frac{\partial u}{\partial x} + v\frac{\partial(\bar{U}+u)}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ &\qquad -\frac{N}{R_m}(B_o + b^v)\left(\frac{\partial b^v}{\partial x} - \frac{\partial b^u}{\partial y}\right) + \frac{N}{R_m}b^v\frac{\partial \bar{b}}{\partial y} \\ &\frac{\partial v}{\partial t} + (\bar{U}+u)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\ &\qquad +\frac{N}{R_m}(\bar{b} + b^u)\left(\frac{\partial b^v}{\partial x} - \frac{\partial b^u}{\partial y}\right) - \frac{N}{R_m}b^u\frac{\partial \bar{b}}{\partial y} \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &\frac{\partial b^u}{\partial t} + (\bar{U}+u)\frac{\partial b^u}{\partial x} + v\frac{\partial(\bar{b}+b^u)}{\partial y} = \frac{1}{R_m}\left(\frac{\partial^2 b^u}{\partial x^2} + \frac{\partial^2 b^u}{\partial y^2}\right) \\ &\qquad +(\bar{b}+b^u)\frac{\partial u}{\partial x} + (B_o + b^v)\frac{\partial u}{\partial y} + b^v\frac{\partial U}{\partial y} \\ &\frac{\partial b^v}{\partial t} + (\bar{U}+u)\frac{\partial b^v}{\partial x} + v\frac{\partial b^v}{\partial y} = \frac{1}{R_m}\left(\frac{\partial^2 b^v}{\partial x^2} + \frac{\partial^2 b^v}{\partial y^2}\right) \\ &\qquad +(\bar{b}+b^u)\frac{\partial v}{\partial x} + (B_o + b^v)\frac{\partial v}{\partial y} \\ &\frac{\partial b^u}{\partial x} + \frac{\partial b^v}{\partial y} = 0. \end{aligned}$$

with initial conditions $u(x,y,0) = u_o(x,y)$, $v(x,y,0) = v_o(x,y)$, $b^u(x,y,0) = b^u_o(x,y)$, $b^v(x,y,0) = b^v_o(x,y)$ for $0 < x < L$, $-1 < y < 1$ and $t > 0$.

V. Energy Analysis

Choosing the energy function

$$E(\mathbf{v},\mathbf{B}) = \int_{-1}^{1}\int_0^L (u^2 + v^2 + b^{u^2} + b^{v^2})\,dxdy \qquad (20)$$

we can compute

$$\dot{E}(\mathbf{v},\mathbf{B}) = 2\int_{-1}^{1}\int_0^L (uu_t + vv_t + b^u b^u_t + b^v b^v_t)\,dxdy. \quad (21)$$

We assume periodic boundary conditions in the streamwise direction, i.e. $\mathbf{v}(x=0) = \mathbf{v}(x=L)$, $\mathbf{B}(x=0) = \mathbf{B}(x=L)$ and $P(x=0) = P(x=L)$. In addition we apply control in the wall normal direction

$$\begin{aligned} u(x,-1,t) &= u(x,1,t) = 0 \\ v(x,-1,t) &= v(x,1,t) = v_{wall}(x,t), \end{aligned}$$

and measure the wall normal component of the induced magnetic field

$$\begin{aligned} b^u(x,-1,t) &= b^u(x,1,t) = 0 \\ b^v(x,-1,t) &= b^v_{bot\_wall}(x,t), b^v(x,1,t) = b^v_{top\_wall}(x,t). \end{aligned}$$

Lemma 1: Taking into account our boundary conditions, the time derivative of $E(\mathbf{v},\mathbf{B})$ along the trajectories can be written as

$$\dot{E}(\mathbf{v},\mathbf{B}) = -\frac{1}{R}m(\mathbf{v},\mathbf{B}) - \int_0^L v_{wall}\left[\Delta p + \frac{\Delta\left[(b^v)^2\right]}{2}\right]dx + g(\mathbf{v},\mathbf{B}), \quad (22)$$

where

$$m(\mathbf{v},\mathbf{B}) = \int_{-1}^1\int_0^L (u_x^2+u_y^2+v_x^2+v_y^2)dxdy + \frac{R}{R_m}\int_{-1}^1\int_0^L \left((b^u_x)^2+(b^u_y)^2+(b^v_x)^2+(b^v_y)^2\right)dxdy, \quad (23)$$

$$\begin{aligned} g(\mathbf{v},\mathbf{B}) &= -\int_{-1}^1\int_0^L \bar{U}' uvdxdy \\ &- \int_{-1}^1\int_0^L \bar{b}' b^u v dxdy \\ &+ \int_{-1}^1\int_0^L \bar{U}' b^u b^v dxdy \\ &+ +\int_{-1}^1\int_0^L \frac{N}{R_m}\bar{b}'\left(ub^v - vb^u\right)dxdy \\ &+ \int_{-1}^1\int_0^L \frac{N}{R_m}\bar{b}\left(b^v_x - b^u_y\right)vdxdy \\ &- \int_{-1}^1\int_0^L \frac{N}{R_m}B_o\left(b^v_x - b^u_y\right)udxdy \\ &+ \int_{-1}^1\int_0^L \bar{b}\left(b^u u_x + b^v v_x\right)dxdy \\ &+ \int_{-1}^1\int_0^L B_o\left(b^u u_y + b^v v_y\right)dxdy \\ &+ \int_{-1}^1\int_0^L \frac{N}{R_m}b^u\left(b^v_x - b^u_y\right)vdxdy \\ &- \int_{-1}^1\int_0^L \frac{N}{R_m}b^v\left(b^v_x - b^u_y\right)udxdy \\ &+ \int_{-1}^1\int_0^L b^u b^u u_x dxdy \\ &+ \int_{-1}^1\int_0^L b^u b^v v_x dxdy \\ &+ \int_{-1}^1\int_0^L b^v b^u u_y dxdy \\ &+ \int_{-1}^1\int_0^L b^v b^v v_y dxdy, \end{aligned} \quad (24)$$

and

$$\begin{aligned} \Delta p &= p(x,1,t) - p(x,-1,t) \\ &= P(x,1,t) - P(x,-1,t) \end{aligned} \quad (25)$$

$$\begin{aligned} \Delta\left[(b^v)^2\right] &= (b^v(x,1,t))^2 - (b^v(x,-1,t))^2 \\ &= {b^v}^2(x,1,t) - {b^v}^2(x,-1,t). \end{aligned} \quad (26)$$

The stretching of material elements accompanied by folding are keys to effective mixing. The measure (23) seems to be strongly connected to mixing since there is a direct relation between stretching of material elements and the spatial gradients of the flow field. Folding is present implicitly in (23) due to the boundedness of the flow domain and the fact that $\mathbf{v}$ satisfies the Navier-Stokes equation.

Lemma 2: The function $g(\mathbf{v},\mathbf{B})$ satisfies

$$|g(\mathbf{v},\mathbf{B})| \le g_1 m(\mathbf{v},\mathbf{B}) + g_2 m^2(\mathbf{v},\mathbf{B}) + g_3\int_0^L v_{wall}^2 dx + g_4\int_0^L (b^v_{top\_wall})^2 dx + g_5\left(\int_0^L (b^v_{top\_wall})^2 dx\right)^2$$

where g_1, g_2, g_3, g_4 and g_5 are constants conveniently defined.

The design goal is a feedback control law, in terms of suction and blowing of fluid normally to the channel wall, that is optimal with respect to some meaningful cost functional related to $m(\mathbf{v},\mathbf{B})$.

Theorem 1: The cost functional

$$J(v_{wall}) = \lim_{t\to\infty}\left[2\beta E(\mathbf{v}(t),\mathbf{B}(t)) + \int_0^t h(\mathbf{v}(\tau),\mathbf{B}(\tau))d\tau\right]$$

where

$$h(\mathbf{v},\mathbf{B}) = \frac{2\beta}{R}m(\mathbf{v},\mathbf{B}) - 2\beta\left[g(\mathbf{v},\mathbf{B}) + g_4\int_0^L (b^v_{top\_wall})^2 dx + g_5\left(\int_0^L (b^v_{top\_wall})^2 dx\right)^2\right] - \beta\int_0^L v_{wall}^2 dx - \beta\int_0^L\left[\Delta p + \frac{\Delta\left[(b^v)^2\right]}{2}\right]^2 dx \quad (27)$$

is maximized by the control

$$v_{wall} = -\left[\Delta p + \frac{\Delta\left[(b^v)^2\right]}{2}\right]. \quad (28)$$

Moreover, solutions of the system described by the dimensionless perturbation equations satisfy

$$h(\mathbf{v},\mathbf{B}) \le k_1 m(\mathbf{v},\mathbf{B}) + k_2 m^2(\mathbf{v},\mathbf{B}) - k_3\int_0^L v_{wall}^2 dx - \beta\int_0^L\left[\Delta p + \frac{\Delta\left[(b^v)^2\right]}{2}\right]^2 dx \quad (29)$$

for arbitrary values of control v_{wall} and with

$$k_1 = 2\beta\left(\frac{1}{R} + g_1\right), \quad k_2 = 2\beta g_2, \quad k_3 = \beta - g_3. \quad (30)$$

Proof. By Lemma 1, we can write equation (27) as

$$h(\mathbf{v},\mathbf{B}) = -2\beta\dot{E}(\mathbf{v},\mathbf{B}) - 2\beta g_4 \int_0^L (b^v_{top\_wall})^2 dx$$
$$-2\beta g_5 \left(\int_0^L (b^v_{top\_wall})^2 dx\right)^2$$
$$-\beta \int_0^L \left(v_{wall} + \left[\Delta p + \frac{\Delta\left[(b^v)^2\right]}{2}\right]\right)^2 dx. \quad (31)$$

In addition, the cost functional can be written as

$$J(v_{wall}) = 2\beta E(\mathbf{v}(0),\mathbf{B}(0))$$
$$-2\beta \lim_{t\to\infty}\left[g_4 \int_0^t \int_0^L (b^v_{top\_wall})^2 dx d\tau\right.$$
$$\left.-g_5 \int_0^t \left(\int_0^L (b^v_{top\_wall})^2 dx\right)^2 d\tau\right]$$
$$-\beta \lim_{t\to\infty} \int_0^t \int_0^L \left(v_{wall} + \left[\Delta p + \frac{\Delta\left[(b^v)^2\right]}{2}\right]\right)^2 dx d\tau.$$

The cost functional is maximized when the last integral is zero. Therefore the control (28) is optimal. In addition, replacing the expression for $\dot{E}(\mathbf{v},\mathbf{B})$ given by Lemma 1 in equation (31) and using Lemma 2 we can write

$$h(\mathbf{v},\mathbf{B}) \le k_1 m(\mathbf{v},\mathbf{B}) + k_2 m^2(\mathbf{v},\mathbf{B})$$
$$-k_3 \int_0^L v_{wall}^2 dx - \beta \int_0^L \left[\Delta p + \frac{\Delta\left[(b^v)^2\right]}{2}\right]^2 dx.$$

□

Inequality (29) implies that $h(\mathbf{v},\mathbf{B})$ cannot be made large without making $m(\mathbf{v},\mathbf{B})$ large as long as β is chosen to make $k_3 > 0$. Thus, the control law (28) maximizes mixing, with minimal control (v_{wall}) and sensing $\left(\Delta p, \Delta\left[(b^v)^2\right]\right)$ effort.

VI. Future Work

The proposed control law will be tested in numerical simulations. The development of a code for the full set of MHD equations will give us not only the possibility of testing the control law proposed in this work but also the opportunity of exploring new ways to suppress or enhance turbulence through both sensing and actuation of the induced magnetic field. The idea of influencing the induced magnetic field modifying its boundary conditions through small magnets embedded into the walls would lead us to an effective way to mitigate or enhance the stabilizing effect of the imposed magnetic field. The possibility of stabilizing the MHD flow using a weaker magnetic field conveniently modulated by feedback instead of using a strong static magnetic field would have a direct impact in hypersonic flight; the heavy magnets could be replaced then by lighter magnets.

The multi-scale complexity of the full set of MHD equations can be eliminated using the so-called "low magnetic Reynold number approximation". Under this approximation the induced magnetic field is neglected with respect to the imposed static magnetic field and therefore the induction equation is eliminated. Now a Poisson equation is solved to obtain the electric potential distribution that determines the current density which in turn defines the Lorentz force term in the Navier-Stokes equation [14]. Although we lose the possibility of sensing and actuating the induced magnetic field, within this more tractable numerical framework we still have the possibility of exploring very interesting approaches such as the use of electrodes to modify the electric potential and current density which in turn can modify the Lorentz force, increasing the air-breaking in hypersonic flights or decreasing the pressure drop in fusion reactor liquid-metal blankets, and the stability property of the system.

VII. References

[1] O. M. Aamo and M. Krstic, *Flow Control by Feedback*, Springer, 2002.

[2] N. B. Morley, S. Smolentsev, L. Barleon, I. R. Kirillov and M. Takahashi, "Liquid magnetohydrodynamics – recent progress and future directions for fusion," *Fusion Engineering and Design*, 51-52, pp. 701-713, 2000.

[3] J. Poggie and D. V. Gaitonde, "Magnetic control of flow past a blunt body: Numerical validation and exploration," *Physics of Fluids*, vol. 14, no. 5, p. 1720, May 2002.

[4] J. Poggie and D. V. Gaitonde, "Computational Studies of Magnetic Control in Hypersonic Flow," AIAA 2001-0196, *39th Aerospace Sciences Meeting and Exhibit*, Reno, NV, 8-11 January 2001.

[5] J-F. Dietiker and K. A. Hoffmann, "Boundary Layer Control in Magnetohydrodynamic Flows," AIAA 2002-0130, *40th Aerospace Sciences Meeting and Exhibit*, Reno, NV, 14-17 January 2002.

[6] J. Poggie and D. V. Gaitonde, "Electrode Boundary Conditions in Magnetogasdynamic Flow Control," AIAA 2002-0199, *40th Aerospace Sciences Meeting and Exhibit*, Reno, NV, 14-17 January 2002.

[7] D. V. Gaitonde and J. Poggie, "Elements of a Numerical Procedure for 3-D MGD Flow Control Analysis," AIAA 2002-0198, *40th Aerospace Sciences Meeting and Exhibit*, Reno, NV, 14-17 January 2002.

[8] A. Balogh, W.-J. Liu and M. Krstic, "Stability enhancement by boundary control in 2D channel flow," *IEEE Transactions on Automatic Control*, vol. 46, pp. 1696-1711, 2001.

[9] O. M. Aamo, M. Krstic and T. R. Bewley, "Control of Mixing by Boundary Feedback in 2D Channel Flow," submitted to *Automatica*.

[10] O. M. Aamo, A. Balogh and M. Krstic, "Optimal mixing by feedback in pipe flow," *15th IFAC World Congress on Automatic Control*, Barcelona, Spain, 2002.

[11] A. Balogh, O. M. Aamo and M. Krstic, "Optimal mixing enhancement in 3D pipe flow," submitted to *IEEE Transactions on Control Systems Technology*.

[12] J. Hartmann, "Theory of the laminar flow of an electrically conductive liquid in a homogeneous magnetic field," *Det Kgl. Danske Videnskabernes Selskab. Mathematisk-fysiske Meddelelser.* **XV** (6), 1-27.

[13] U. Müller and L. Bühler, *Magnetofluiddynamics in Channels and Containers*, Springer, 2001.

[14] D. Lee and H. Choi, "Magnetohydrodynamic turbulent flow in a channel at low magnetic Reynolds number," *Journal on Fluid Mechanics*, vol. 439, pp. 367-394, 2001.