Nonlinear Boundary Control of Non-Burning Plasma Kinetic Profiles in Cylindrical Geometry †

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Abstract

The control of kinetic profiles is among the most important problems in fusion reactor research. It is strongly related to a great number of other problems in fusion energy generation such as burn control, transport reduction, confinement time improvement, MHD instability avoidance and high-β or high-confinement operating modes access.

We consider in this work a set of nonlinear partial differential equations (PDE's) describing approximately the dynamics of the density and energy profiles in a non-burning plasma. Applying a backstepping design we control the kinetic profiles by means of thermal and density actuation at the boundary. Numerical simulations show that the feedback control law designed using only one step of backstepping can successfully control the kinetic profiles.

1 Introduction

The regulation of the kinetic and current profiles is essential to achieving optimal fusion performance and making fusion an economically viable source of energy. Plasma behavior is critically influenced by the plasma density, current density and temperature profiles.

Burn control of an ignited or subignited plasma is directly influenced by the kinetic profiles. A D-T plasma may be thermally unstable in some regions of operation and a tight control is required for avoiding thermal excursion or quenching. Auxiliary heating, fueling and impurity injection are among the most common actuators used to keep the density and temperature of the plasma at a desired working point. Among the problems related to the control of the kinetic variables, the problem of burn control is the most extensive found in the literature. This can be explained by the fact that the problem of controlling the burn instability can be approached considering a 0-D (zero-dimensional) model where spatially averaged quantities are considered. The availability of conventional control tools that are capable of dealing with this kind of model, where the dynamics of the average kinetic variables is described by ODE's, encourages the study of the problem. Recently, we have introduced a new approach where the linearization of the model is avoided and much higher levels of performance and robustness are achieved [1, 2]. However, the 0-D control of the burn instability using modulation of bulk heating, fueling and impurity density does not take into account the 1-D (one-dimensional) effect of this modulation on the profiles. The heating, fueling and impurity density are distributed throughout the plasma volume affecting the density and temperature profiles which in turn can change the transport mode, the energy confinement time and the plasma stability.

We need a control technique that can deal with the distributed and nonlinear nature of those quantities, their coupling with one another, and their, at times conflicting, control objectives. The work we present here is inspired by [3, 4, 5] and to some extent by [6, 7]. In all these mentioned works the 1-D model is represented by a set of nonlinear PDE's. The reduction of the distributed parameter description of the system to a lumped parameter description is carried out using different methods. The resulting set of ODE's are linearized and conventional linear control methods are applied for the synthesis of the controller. In contrast to these previous works, the control method presented in this paper is based on the full nonlinear model. As we showed for the 0-D case, the plasma dynamics is highly nonlinear and fundamental information about the system is lost through the linearization, imposing in this way a limit on operability. Therefore, the linearization of the model should be avoided, and this is central to our approach. We control the system by means of thermal and density actuation.

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The goal of the controller is to make the kinetic profiles converge to their desired equilibrium profiles. We are interested in constructing a stabilizing controller that achieves stability for unstable equilibrium profiles and increases performance for stable equilibrium profiles. In order to simplify this initial approach to kinetic profile control in fusion reactors, we consider a non-burning plasma whose dynamics is described by a 1-D nonlinear PDE model. This 1-D nonlinear PDE model consists of the diffusion equations of the kinetic variables in cylindrical geometry where the diffusion coefficients, on the other hand, are nonlinear functions of these kinetic variables. The original set of PDE's is discretized in space using a finite difference method which gives a high order set of coupled nonlinear ODE's. Applying a backstepping design we obtain a discretized coordinate transformation that transforms the original system into a properly chosen target system that is asymptotically stable in l^2-norm. To achieve such stability for the target system, convenient boundary conditions are chosen. Then, using the property that the discretized coordinate transformation is invertible for an arbitrary (finite) grid choice, we conclude that the discretized version of the original system is asymptotically stable and obtain a nonlinear feedback boundary control law for the energy and density in the original set of coordinates. This technique has been already applied successfully for other different physical applications [8, 9].

The paper is organized as follows. In Section 2 a nonlinear one dimensional PDE model that governs the dynamics of the density and energy profiles in a non-burning plasma is introduced. The control objective is stated in Section 3. In Section 4 a nonlinear feedback control law that achieves asymptotic stabilization is presented, followed by the proof of stability for the target system in Section 5. A feedback control law designed on a coarse grid is shown through a simulation study to successfully control the kinetic profiles of the plasma in Section 6. Finally, some conclusions and suggestions are stated in Section 7.

2 Model

The mathematical model used in this work is basically the set of transport equations in cylindrical geometry used in [3]. The energy and density transport equations are given by

$$\frac{3}{2}\frac{\partial nT}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(\kappa\frac{\partial T}{\partial r} + \frac{3}{2}DT\frac{\partial n}{\partial r}\right) - P_{br} + P_{aux}, \quad (1)$$

$$\frac{\partial n}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial n}{\partial r} - nV_p\right) + S, \quad (2)$$

where n is the density, T is the temperature, $E = \frac{3}{2}nT$ is the energy, P_{aux} is the auxiliary heating power (actuator) and S is the fueling rate (actuator). The radiation loss considered in this model is the bremsstrahlung loss

$$P_{br} = A_b Z_{eff} n_e^2 \sqrt{T}$$

where $Z_{eff} = (\sum_i n_i Z_i^2)/n_e$, n_e is the electron density and n_i is the ion density. Since this model describes a non-burning plasma the alpha particle density is neglected. Therefore the quasineutrality condition $n_e = n_i Z_i$ implies that $n_e = n_i$ because the only ion present in the plasma is the deuterium-tritium ion ($Z_i = 1$). This implies in turn that $Z_{eff} = 1$. The electron and ion temperatures are considered to be equal. In addition, $\kappa = n\chi$ is the heat conduction coefficient, $\chi = \frac{n(0)}{n(r)}\frac{m^2}{s}$ is the thermal diffusivity coefficient, $V_p = \frac{1}{2}\frac{D}{T}\frac{\partial T}{\partial r}$ is the inward pinch velocity and D is the diffusion coefficient. With the purpose of simplification, we take $D = \frac{2}{3}\chi$, which is an approximation of the diffusion coefficient used in [3]. This approximation is not a requirement for the control method and its only purpose is the simplification of this presentation. In this way, we can reduce equations (1) and (2) to

$$\frac{\partial E}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial E}{\partial r}\right] - P_{br} + P_{aux}, \quad (3)$$

$$\frac{\partial n}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(D\frac{\partial n}{\partial r} - nV_p\right)\right] + S. \quad (4)$$

We consider the following arbitrary boundary conditions; $E_r(r=0) = 0$, $n_r(r=0) = 0$, $E_r(r=a) = k_E E(a)$ and $n_r(r=a) = k_n n(a)$.

3 Control Objective

We write $E(r,t) = \bar{E}(r) + \tilde{E}(r,t)$ and $n(r,t) = \bar{n}(r) + \tilde{n}(r,t)$, where $\bar{E}(r)$ and $\bar{n}(r)$ are the equilibrium profiles which in turn are the solutions of the equilibrium equations

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\bar{D}\frac{\partial \bar{E}}{\partial r}\right] - \bar{P}_{br} + \bar{P}_{aux}, \quad (5)$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\bar{D}\frac{\partial \bar{n}}{\partial r} - \bar{n}\bar{V}_p\right)\right] + \bar{S}. \quad (6)$$

with boundary conditions $\bar{E}_r(r=0) = 0$, $\bar{n}_r(r=0) = 0$, $\bar{E}_r(r=a) = k_E\bar{E}(a)$ and $\bar{n}_r(r=a) = k_n\bar{n}(a)$. Proper selection of the boundary conditions and the equilibrium profiles for the auxiliary power $\bar{P}_{aux}$ and fueling rate $\bar{S}$ allows us to achieve the desired equilibrium profiles for the energy and the density. It is important to note that in this approach to kinetic profile control we consider only density and thermal actuation at the edge of the plasma. Therefore, the fueling rate $S = \bar{S}$ and the auxiliary power $P_{aux} = \bar{P}_{aux}$ are used only for the definition of the equilibrium profiles. The dynamics of the deviation variables $\tilde{E}(r,t)$ and $\tilde{n}(r,t)$ is given by

$$\frac{\partial \tilde{E}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial(\bar{E}+\tilde{E})}{\partial r}\right] - P_{br} + P_{aux}$$

$$
\begin{aligned}
&= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \bar{E}}{\partial r}\right] - P_{br} + P_{aux}, \\
\frac{\partial \tilde{n}}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(D\frac{\partial(\bar{n}+\tilde{n})}{\partial r} - nV_p\right)\right] + S \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{3}{2}\frac{\partial \tilde{n}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{E}}{\partial r}\right)\right] \\
&\quad + \frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{3}{2}\frac{\partial \bar{n}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \bar{E}}{\partial r}\right)\right] + S,
\end{aligned}
$$

where we have written

$$
\begin{aligned}
V_p &= \frac{1}{2}\frac{D}{T}\frac{\partial T}{\partial r} = \frac{1}{2}D\left(\frac{1}{E}\frac{\partial E}{\partial r} - \frac{1}{n}\frac{\partial n}{\partial r}\right) \\
&= \frac{1}{2}D\left(\frac{1}{E}\frac{\partial \bar{E}}{\partial r} - \frac{1}{n}\frac{\partial \bar{n}}{\partial r}\right) + \frac{1}{2}D\left(\frac{1}{E}\frac{\partial \tilde{E}}{\partial r} - \frac{1}{n}\frac{\partial \tilde{n}}{\partial r}\right).
\end{aligned}
$$

Defining

$$
g(E,n) = \frac{\partial}{\partial r}\left[D\frac{\partial \bar{E}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \bar{E}}{\partial r} - P_{br} + P_{aux}, \quad (7)
$$

$$
\begin{aligned}
f(E,n) &= \frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial \bar{n}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \bar{n}}{\partial r}\right\} \quad (8) \\
&\quad - \frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial \bar{E}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial \bar{E}}{\partial r}\right\} + S,
\end{aligned}
$$

we rewrite the equations for the deviation variables as

$$
\frac{\partial \tilde{E}}{\partial t} = \frac{\partial}{\partial r}\left[D\frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{E}}{\partial r} + g(E,n), \quad (9)
$$

$$
\begin{aligned}
\frac{\partial \tilde{n}}{\partial t} &= \frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial \tilde{n}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{n}}{\partial r}\right\} \quad (10) \\
&\quad - \frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial \tilde{E}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial \tilde{E}}{\partial r}\right\} + f(E,n),
\end{aligned}
$$

with boundary conditions $\tilde{E}_r(r=0) = 0$, $\tilde{n}_r(r=0) = 0$, $\tilde{E}_r(r=a) = k_E\tilde{E}(a) + \Delta\tilde{E}_r$ and $\tilde{n}_r(r=a) = k_n\tilde{n}(a) + \Delta\tilde{n}_r$. The objective is to stabilize $\tilde{E}(r,t)$ and $\tilde{n}(r,t)$, making them converge to zero, by using $\Delta\tilde{E}_r(t)$ and $\Delta\tilde{n}_r(t)$ as actuation at the edge of the plasma.

4 Controller Design

Figure 1 summarizes the essence of the control method. We dicretize the original set of PDE's in space using a finite difference method which gives a high order set of coupled nonlinear ordinary differential equations (ODE's). Applying a backstepping design we obtain a discretized coordinate transformation that transforms the original system into a properly chosen target system that is asymptotically stable in l^2-norm. To achieve such stability for the target system, convenient boundary conditions are chosen. Then, using the property that the discretized coordinate transformation is invertible for an arbitrary (finite) grid choice, we conclude that the discretized version of the original system is asymptotically stable and obtain a nonlinear feedback boundary control law for the energy and density in the original set of coordinates. The idea is to design controllers using only a small number of steps of backstepping, or equivalently using only a small number of state measurements. The measurements are taken from the core of the plasma and the actuation is applied at the edge of the plasma.

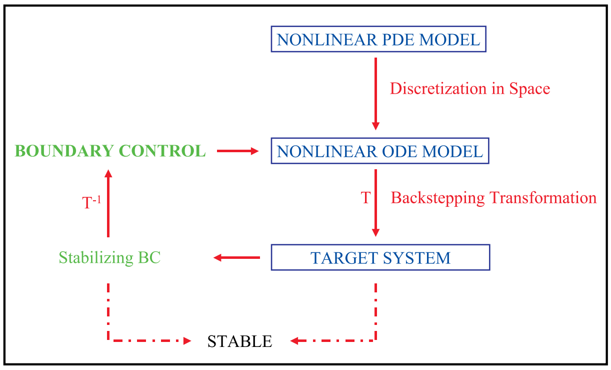

Figure 1: Control Method Scheme

To discretize the problem, let us start by defining $h = \frac{1}{N}$, where N is an integer. Then using the notation $x_i(t) = x(ih,t)$, $i = 0, 1, ..., N$, we write the discretized version of equations (9)-(10) as

$$
\begin{aligned}
\dot{\tilde{E}}_i &= \frac{D_{i+\frac{1}{2}}\tilde{E}_{i+1} - 2D_i\tilde{E}_i + D_{i-\frac{1}{2}}\tilde{E}_{i-1}}{h^2} \quad (11) \\
&\quad + \frac{1}{ih}D_i\frac{\tilde{E}_{i+1} - \tilde{E}_i}{h} + g_i, \\
g_i &= \frac{D_{i+\frac{1}{2}}\bar{E}_{i+1} - 2D_i\bar{E}_i + D_{i-\frac{1}{2}}\bar{E}_{i-1}}{h^2} \quad (12) \\
&\quad + \frac{1}{ih}D_i\frac{\bar{E}_{i+1} - \bar{E}_i}{h} - (P_{br})_i + (\bar{P}_{aux})_i, \\
(P_{br})_i &= A_b n_i^2\sqrt{\frac{2}{3}\frac{E_i}{n_i}},
\end{aligned}
$$

$$
\begin{aligned}
\dot{\tilde{n}}_i = \frac{3}{2}&\left\{\frac{D_{i+\frac{1}{2}}\tilde{n}_{i+1} - 2D_i\tilde{n}_i + D_{i-\frac{1}{2}}\tilde{n}_{i-1}}{h^2}\right. \quad (13) \\
&\left. + \frac{1}{ih}D_i\frac{\tilde{n}_{i+1} - \tilde{n}_i}{h}\right\} \\
- \frac{1}{2}&\left\{\frac{\left(\frac{Dn}{E}\right)_{i+\frac{1}{2}}\tilde{E}_{i+1} - 2\left(\frac{Dn}{E}\right)_i\tilde{E}_i + \left(\frac{Dn}{E}\right)_{i-\frac{1}{2}}\tilde{E}_{i-1}}{h^2}\right. \\
&\left. + \frac{1}{ih}\left(\frac{Dn}{E}\right)_i\frac{\tilde{E}_{i+1} - \tilde{E}_i}{h}\right\} + f_i,
\end{aligned}
$$

$$
\begin{aligned}
f_i = \frac{3}{2}&\left\{\frac{D_{i+\frac{1}{2}}\bar{n}_{i+1} - 2D_i\bar{n}_i + D_{i-\frac{1}{2}}\bar{n}_{i-1}}{h^2}\right. \quad (14) \\
&\left. + \frac{1}{ih}D_i\frac{\bar{n}_{i+1} - \bar{n}_i}{h}\right\}
\end{aligned}
$$

$$- \frac{1}{2}\left\{\frac{\left(\frac{Dn}{E}\right)_{i+\frac{1}{2}}\bar{E}_{i+1} - 2\left(\frac{Dn}{E}\right)_i \bar{E}_i + \left(\frac{Dn}{E}\right)_{i-\frac{1}{2}}\bar{E}_{i-1}}{h^2}\right.$$
$$\left. + \frac{1}{ih}\left(\frac{Dn}{E}\right)_i \frac{\bar{E}_{i+1} - \bar{E}_i}{h}\right\} + \bar{S}_i,$$

with boundary conditions $(\tilde{E}_1 - \tilde{E}_0)/h = 0$, $(\tilde{n}_1 - \tilde{n}_0)/h = 0$, $(\tilde{E}_N - \tilde{E}_{N-1})/h = k_E\tilde{E}_N + \Delta\tilde{E}_r$, $(\tilde{n}_N - \tilde{n}_{N-1})/h = k_n\tilde{n}_N + \Delta\tilde{n}_r$ and where $D_{i-\frac{1}{2}} = \frac{1}{2}D_i + \frac{1}{2}D_{i-1}$, $D_{i+\frac{1}{2}} = \frac{3}{2}D_i - \frac{1}{2}D_{i-1}$, $\left(\frac{Dn}{E}\right)_{i-\frac{1}{2}} = \frac{1}{2}\left(\frac{Dn}{E}\right)_i + \frac{1}{2}\left(\frac{Dn}{E}\right)_{i-1}$, $\left(\frac{Dn}{E}\right)_{i+\frac{1}{2}} = \frac{3}{2}\left(\frac{Dn}{E}\right)_i - \frac{1}{2}\left(\frac{Dn}{E}\right)_{i-1}$.

We consider now the asymptotically stable (in L^2 norm) target system

$$\begin{aligned}
\frac{\partial \tilde{F}}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{F}}{\partial r}\right] - C_F\tilde{F} \\
&= \frac{\partial}{\partial r}\left[D\frac{\partial \tilde{F}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{F}}{\partial r} - C_F\tilde{F}, \qquad (15)\\
\frac{\partial \tilde{m}}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{3}{2}\frac{\partial \tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{F}}{\partial r}\right)\right] - C_m\tilde{m} \\
&= \frac{3}{2}\left\{\frac{\partial}{\partial r}\left[D\frac{\partial \tilde{m}}{\partial r}\right] + \frac{1}{r}D\frac{\partial \tilde{m}}{\partial r}\right\} \qquad (16)\\
&\quad -\frac{1}{2}\left\{\frac{\partial}{\partial r}\left[\frac{Dn}{E}\frac{\partial \tilde{F}}{\partial r}\right] + \frac{1}{r}\frac{Dn}{E}\frac{\partial \tilde{F}}{\partial r}\right\} - C_m\tilde{m},
\end{aligned}$$

where $C_F > 0$, $C_m > 0$ and the boundary conditions are given by $\tilde{F}_r(r=0) = 0$, $\tilde{m}_r(r=0) = 0$, $\tilde{F}_r(r=a) = -G\tilde{F}(a)$ and $\tilde{m}_r(r=a) = -G\tilde{m}(a)$, with $G > 0$. We write the discretized equations for the target system as

$$\begin{aligned}
\dot{\tilde{F}}_i &= \frac{D_{i+\frac{1}{2}}\tilde{F}_{i+1} - 2D_i\tilde{F}_i + D_{i-\frac{1}{2}}\tilde{F}_{i-1}}{h^2} \qquad (17)\\
&\quad + \frac{1}{ih}D_i\frac{\tilde{F}_{i+1} - \tilde{F}_i}{h} - C_F\tilde{F}_i,\\
\dot{\tilde{m}}_i &= \frac{3}{2}\left\{\frac{D_{i+\frac{1}{2}}\tilde{m}_{i+1} - 2D_i\tilde{m}_i + D_{i-\frac{1}{2}}\tilde{m}_{i-1}}{h^2}\right. \qquad (18)\\
&\quad \left. + \frac{1}{ih}D_i\frac{\tilde{m}_{i+1} - \tilde{m}_i}{h}\right\}\\
&\quad -\frac{1}{2}\left\{\frac{\left(\frac{Dn}{E}\right)_{i+\frac{1}{2}}\tilde{F}_{i+1} - 2\left(\frac{Dn}{E}\right)_i\tilde{F}_i + \left(\frac{Dn}{E}\right)_{i-\frac{1}{2}}\tilde{F}_{i-1}}{h^2}\right.\\
&\quad \left. + \frac{1}{ih}\left(\frac{Dn}{E}\right)_i\frac{\tilde{F}_{i+1} - \tilde{F}_i}{h}\right\} - C_m\tilde{m}_i,
\end{aligned}$$

with boundary conditions written as $(\tilde{F}_1 - \tilde{F}_0)/h = 0$, $(\tilde{m}_1 - \tilde{m}_0)/h = 0$, $(\tilde{F}_N - \tilde{F}_{N-1})/h = -G\tilde{F}_N$, $(\tilde{m}_N - \tilde{m}_{N-1})/h = -G\tilde{m}_N$. Finally we look for a backstepping transformation of the discretized original system into the discretization of the target system. This coordinate transformation is sought in the form

$$\begin{aligned}
\tilde{F}_i &= \tilde{E}_i - \alpha_{i-1}(\tilde{E}_1, ..., \tilde{E}_{i-1}, \tilde{n}_1, ..., \tilde{n}_{i-1}), \qquad (19)\\
\tilde{m}_i &= \tilde{n}_i - \beta_{i-1}(\tilde{E}_1, ..., \tilde{E}_{i-1}, \tilde{n}_1, ..., \tilde{n}_{i-1}). \qquad (20)
\end{aligned}$$

Considering that $\dot{\alpha}_{i-1} = \dot{\tilde{E}}_i - \dot{\tilde{F}}_i$ and $\dot{\beta}_{i-1} = \dot{\tilde{n}}_i - \dot{\tilde{m}}_i$, using equations (11)-(13) and (17)-(18), and taking into account the definitions (19)-(20), we can obtain

$$\begin{aligned}
\alpha_i &= \frac{1}{D_{i+\frac{1}{2}} + \frac{D_i}{i}}\left[\left(2D_i + \frac{D_i}{i} + C_Fh^2\right)\alpha_{i-1}\right.\\
&\quad \left. - D_{i-\frac{1}{2}}\alpha_{i-2} - h^2g_i - h^2C_F\tilde{E}_i + h^2\dot{\alpha}_{i-1}\right], \qquad (21)\\
\beta_i &= \frac{1}{\frac{3}{2}\left(D_{i+\frac{1}{2}} + \frac{D_i}{i}\right)}\left\{\left[\frac{3}{2}\left(2D_i + \frac{D_i}{i}\right) + C_mh^2\right]\beta_{i-1}\right.\\
&\quad \left. - \frac{3}{2}D_{i-\frac{1}{2}}\beta_{i-2} - h^2f_i + r_i - h^2C_m\tilde{n}_i + h^2\dot{\beta}_{i-1}\right\}, \qquad (22)
\end{aligned}$$

starting with $\alpha_0 = \beta_0 = 0$ and where

$$\begin{aligned}
r_i &= \frac{1}{2}\left\{\left(\frac{Dn}{E}\right)_{i+\frac{1}{2}}\alpha_i - 2\left(\frac{Dn}{E}\right)_i\alpha_{i-1}\right.\\
&\quad \left. + \left(\frac{Dn}{E}\right)_{i-\frac{1}{2}}\alpha_{i-2} + \frac{1}{i}\left(\frac{Dn}{E}\right)_i(\alpha_i - \alpha_{i-1})\right\}, \qquad (23)\\
\dot{\alpha}_{i-1} &= \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \tilde{E}_k}\dot{\tilde{E}}_k + \sum_{k=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \tilde{n}_k}\dot{\tilde{n}}_k, \qquad (24)\\
\dot{\beta}_{i-1} &= \sum_{k=1}^{i-1}\frac{\partial \beta_{i-1}}{\partial \tilde{E}_k}\dot{\tilde{E}}_k + \sum_{k=1}^{i-1}\frac{\partial \beta_{i-1}}{\partial \tilde{n}_k}\dot{\tilde{n}}_k. \qquad (25)
\end{aligned}$$

Using the equations for the boundary conditions of the discretized original system and the discretized target system, and taking into account the definitions (19)-(20), we can define the controls as

$$\begin{aligned}
\Delta\tilde{E}_r &= \frac{\alpha_{N-1} - \alpha_{N-2}}{h} - k_E\tilde{E}_N - G\left(\tilde{E}_N - \alpha_{N-1}\right), \qquad (26)\\
\Delta\tilde{n}_r &= \frac{\beta_{N-1} - \beta_{N-2}}{h} - k_n\tilde{n}_N - G\left(\tilde{n}_N - \beta_{N-1}\right). \qquad (27)
\end{aligned}$$

These expressions for $\Delta\tilde{E}_r$ and $\Delta\tilde{n}_r$ allows us to finally write the stabilizing laws for the modulation of the energy and the density at the edge of the plasma

$$\begin{aligned}
\tilde{E}_N &= \alpha_{N-1} + \frac{1}{(1+Gh)}\left[\tilde{E}_{N-1} - \alpha_{N-2}\right], \qquad (28)\\
\tilde{n}_N &= \beta_{N-1} + \frac{1}{(1+Gh)}\left[\tilde{n}_{N-1} - \beta_{N-2}\right]. \qquad (29)
\end{aligned}$$

5 Asymptotic Stability of the Discretized Target System

To show stability of the target system (15)-(16), we take the Lyapunov function candidate

$$V = \frac{1}{2}\int_0^a r\left(\frac{\tilde{F}^2}{k^2} + \tilde{m}^2\right)dr$$

with $k = 1.380662\ 10^{-23}\ \frac{J}{K}$. Then we have

$$\begin{aligned}\dot{V} =& \int_0^a r\left(\frac{\tilde{F}}{k^2}\dot{\tilde{F}} + \tilde{m}\dot{\tilde{m}}\right)dr\\ =& \int_0^a r\left(\frac{\tilde{F}}{k^2}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{F}}{\partial r}\right] - C_F\tilde{F}\right\}\right.\\ &+ \left.\tilde{m}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\left(\frac{3}{2}\frac{\partial \tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{F}}{\partial r}\right)\right] - C_m\tilde{m}\right\}\right)dr\\ =& \frac{\tilde{F}}{k^2}rD\frac{\partial \tilde{F}}{\partial r}\bigg|_0^a - \int_0^a \frac{1}{k^2}rD\left(\frac{\partial \tilde{F}}{\partial r}\right)^2 dr\\ &- \frac{C_F}{k^2}\int_0^a r\tilde{F}^2 dr + \tilde{m}rD\left(\frac{3}{2}\frac{\partial \tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{F}}{\partial r}\right)\bigg|_0^a\\ &- \int_0^a \frac{\partial \tilde{m}}{\partial r}rD\left(\frac{3}{2}\frac{\partial \tilde{m}}{\partial r} - \frac{1}{2}\frac{n}{E}\frac{\partial \tilde{F}}{\partial r}\right)dr - C_m\int_0^a r\tilde{m}^2 dr\\ =& aD(a)\left[\frac{\tilde{F}(a)}{k^2}\tilde{F}_r(a) + \frac{3}{2}\tilde{m}(a)\tilde{m}_r(a)\right.\\ &- \left.\frac{1}{2}\frac{n(a)}{E(a)}\tilde{m}(a)\tilde{F}_r(a)\right] - \int_0^a r\left[\frac{C_F}{k^2}\tilde{F}^2 + C_m\tilde{m}^2\right]dr\\ &- \int_0^a r\frac{D}{k^2}\tilde{F}_r^2 dr - \frac{3}{2}\int_0^a rD\tilde{m}_r^2 dr + \frac{1}{3}\int_0^a rD\frac{\tilde{F}_r\tilde{m}_r}{T}dr\end{aligned}$$

Taking into account the boundary conditions for the target system, we have

$$\begin{aligned}\dot{V} =& -\int_0^a r\left[\frac{C_F}{k^2}\tilde{F}^2 + C_m\tilde{m}^2\right]dr - \frac{1}{2}GaD(a)\tilde{m}^2(a)\\ &- \frac{1}{2}\int_0^a rD\tilde{m}_r^2 dr + \int_0^a rD\left\{-\frac{\tilde{F}_r^2}{k^2} - \tilde{m}_r^2 + \frac{1}{3}\frac{\tilde{F}_r\tilde{m}_r}{T}\right\}dr\\ &+ GaD(a)\left[-\frac{\tilde{F}^2(a)}{k^2} - \tilde{m}^2(a) + \frac{1}{3}\frac{\tilde{F}(a)\tilde{m}(a)}{T(a)}\right]\\ \dot{V} \le& -CV + GaD(a)\left[-\frac{\tilde{F}^2(a)}{k^2} - \tilde{m}^2(a) + \frac{|\tilde{F}(a)||\tilde{m}(a)|}{T(a)}\right]\\ &+ \int_0^a rD\left\{-\frac{\tilde{F}_r^2}{k^2} - \tilde{m}_r^2 + \frac{|\tilde{F}_r||\tilde{m}_r|}{T}\right\}dr,\end{aligned}$$

where $C = min(C_F, C_m)$. Writing $T = kT^*$ where T^* is in K (Kelvin), while T is in J (Joule), and taking into account that $T^* >> 1$, we can state

$$\begin{aligned}\dot{V} \le& -CV + GaD(a)\left[-\frac{\tilde{F}^2(a)}{k^2} - \tilde{m}^2(a) + \left|\frac{\tilde{F}(a)}{k}\right||\tilde{m}(a)|\right]\\ &+ \int_0^a rD\left\{-\frac{\tilde{F}_r^2}{k^2} - \tilde{m}_r^2 + \left|\frac{\tilde{F}_r}{k}\right||\tilde{m}_r|\right\}dr.\end{aligned}$$

By Young's inequality we know that

$$\left[-\frac{\tilde{F}^2(a)}{k^2} - \tilde{m}^2(a) + \left|\frac{\tilde{F}(a)}{k}\right||\tilde{m}(a)|\right] \le 0,$$

$$\int_0^a rD\left\{-\frac{\tilde{F}_r^2}{k^2} - \tilde{m}_r^2 + \left|\frac{\tilde{F}_r}{k}\right||\tilde{m}_r|\right\}dr \le 0,$$

and we conclude that $\dot{V} \le -CV$ showing that the system is asymptotically stable.

The proof that the discretized target system (17)-(18) is asymptotically stable in l^2 norm would be completely analogous. The discrete Lyapunov function $V_d = \frac{1}{2}\sum_{i=0}^N\left(\frac{\tilde{F}_i^2}{k^2} + \tilde{m}_i^2\right)$ would be considered instead and following identical procedure the condition $\dot{V}_d \le -CV_d$ would be obtained.

6 Simulation Results

The simulation presented in this section is run using FTCS finite difference method for a time step $\Delta t = 0.001$ sec, $a = 2.4$ m and $N_s = 24 \Rightarrow h_s = 0.1$. The subscript "s" stands for simulation. In this way we differentiate the fine grid used for simulation purposes and the coarse grid used for control design purposes. The controller is designed using only one step of backstepping, i.e. for $N = 2$.

For the considered non-burning plasma, quadratic profiles $\bar{S} = S_0[1-(r/a)^2)]$ and $\bar{P}_{aux} = (P_{aux})_0 * [(1-(r/a)^2]$, the equilibrium profiles given by equations (5)-(6) are stable. However the rate of convergence to the equilibrium profiles from some initially perturbed profiles is very slow. Therefore, the main goal of the controller in this case is the improvement of performance. Considering $\tilde{E}(r,0) = (-1+2*r/a)10^5$ and $\tilde{n}(r,0) = (-1+2*r/a)10^{19}$, figure 2 shows the evolution of the kinetic profiles from these initial perturbed profiles to their equilibrium values. It is possible to note from the figures that the settling time is approximately 2 seconds. This represents an improvement of an order of magnitude with respect to the open loop settling time. Figure 3 show the actuation at the edge of the plasma that makes this possible.

7 Conclusions and Future Work

A nonlinear feedback controller based on Lyapunov backstepping design that achieves asymptotic stabilization of the equilibrium kinetic profiles in a cylindrical plasma reactor has been synthesized. The result holds for any finite discretization in space of the original PDE model. The simulation study shows that the boundary controller designed using only one step of backstepping, i.e. using only one measurement from the interior of the reactor, can successfully control the kinetic profiles.

The control of the kinetic profiles by boundary control has been shown to be feasible. However, more study is necessary to find the way of modulating the kinetic

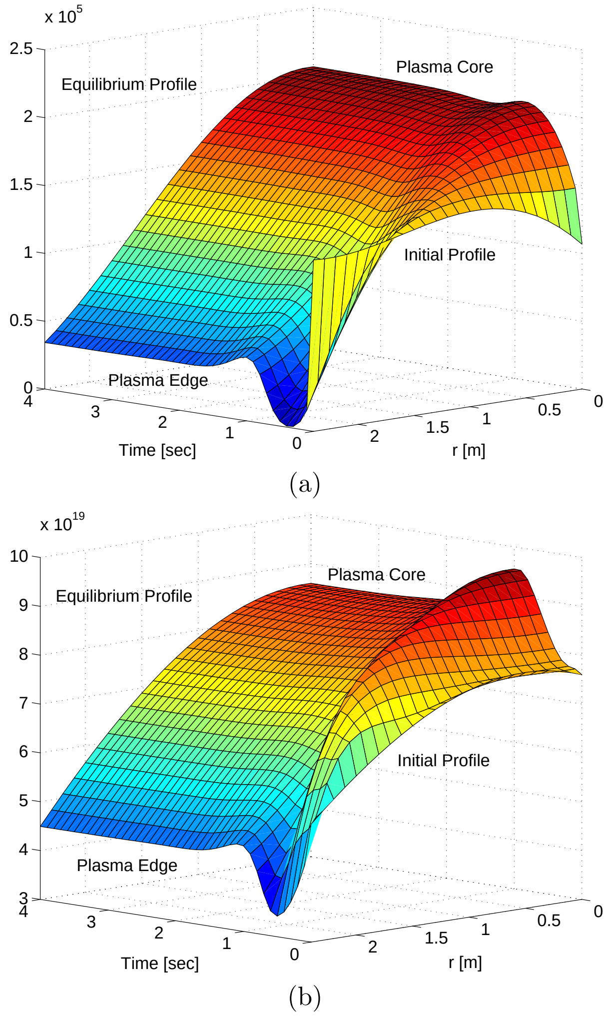

Figure 2: Profile evolution in time for E (a) and n (b).

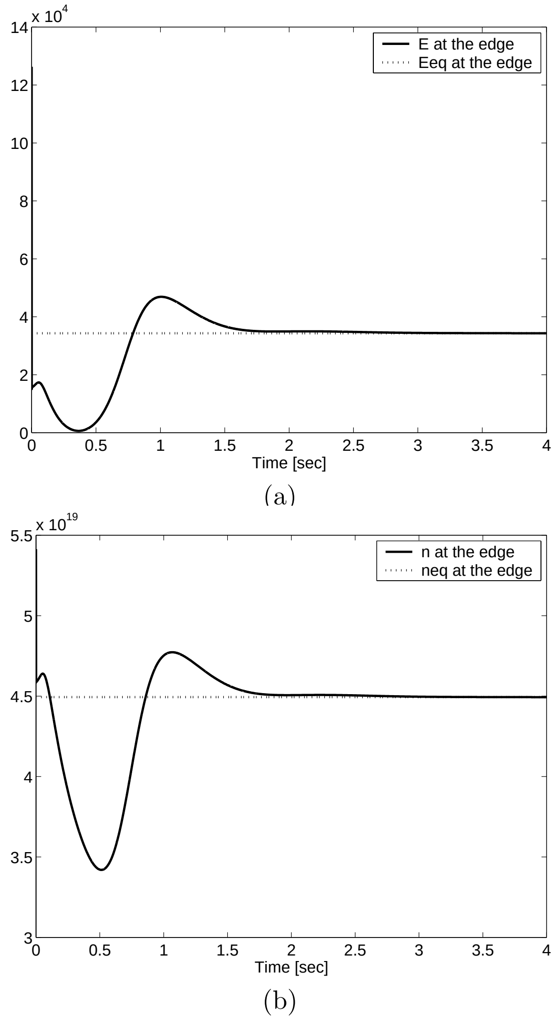

Figure 3: Edge modulation for E (a) and n (b).

variables at the edge of the plasma through the modulation of physical properties of the scrape-off layer. In the future a zero-dimensional model of the tokamak scrape-off layer will be used as a complement of the one-dimensional model for the core. This will allow us to work with more realistic boundary conditions. In this way, we are going to be able not only to search for physical ways to achieve the modulation of the kinetic variables at the edge of the plasma required by our control method but also to work with kinetic profiles which are closer to the ones found in real reactors. In case the necessary modulation of the temperature and density at the edge of the plasma could not be achieved by physical means, actuation directly in the core of the plasma would be considered; approaching in this way a less challenging problem where the auxiliary power and fueling rate is used not only for the definition of the equilibrium profiles but also for the stabilization of such profiles. In addition, in the future a burning plasma (inherently thermally unstable) model and more updated correlations for the physical parameters (D, κ, χ, V_p) will be considered.

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