Extremum Seeking Adaptive Control of Beam Envelope in Particle Accelerators

Eugenio Schuster, Nicholas Torres, Chao Xu

***Abstract*— The matching problem for a low energy transport system in a charged particle accelerator is approached using the extremum seeking feedback method for non-model-based adaptive control. The beam dynamics, used only for simulation purposes, is modeled using the KV (Kapchinsky-Vladimirsky) envelope equations. Extremum seeking is employed for the lens adaptive tuning in both the matching and periodic channels. Numerical simulations illustrate the effectiveness of this approach.**

I. Introduction

In a particle accelerator, charged particles such as electrons, protons, or heavy ions are accelerated by electromagnetic fields to serve as light source (e.g., synchrotron radiation) or to collide with targets. In the last case, as a result of the collision many other subatomic particles are created and detected. From the information collected by the detectors, properties of the particles and their interactions can be determined. Accelerators are used for research in high-energy and nuclear physics, synchrotron radiation, medical therapies, and some industrial applications. The higher the energy of the accelerated particles, the more closely the structure of matter can be probed.

In this work we approach the beam matching problem, where the beam must be matched to the acceptance ellipse of an accelerating structure or transport section. Specifically we consider a fixed-geometry matching section consisting of six quadrupole lenses, an under-determined system. The objective of the matching section is to take any arbitrary initial beam state and "match" it to the acceptance ellipse of the following section (a periodic section, for instance), i.e., any given initial state x_{ini} to a prescribed final state x_{fin}, through the control of the lens strengths in the matching channel. We also consider that the matching section is followed by a fixed-geometry periodic section, where initial and final states are identical (x_p). A review of beam transport including matching was recently presented in [1]. We assume the channels to be composed of discrete beamline elements, such as quadrupole lenses, and drifts. These elements are cascaded along the beam axis, considered the z axis, to form the matching and periodic channels. The matching channel configuration is depicted in Figure 1-a. The inputs to the lenses, labeled θ_1, θ_2, θ_3, θ_4, θ_5, and θ_6 in the figure, represent the focusing strengths of the lenses and are the parameters of the matching channel that may be varied. The periodic channel configuration is depicted in Figure 1-b. The focusing function in this channel must be symmetric, i.e., the absolute values of the lens strengths must be identical and equal to θ_p. This common lens strength is the parameter of the periodic channel that may be varied.

Several codes based on model-based optimization techniques are available to find a matching solution off-line. As a mode of example, in [2] the problem was approached for the six lenses case, as the one considered in this work, using local (nonlinear programming) and global (dynamic programming) methods. The major shortcoming of these methods is their dependence on the model. The accuracy of the calculation is limited by the uncertainties associated with the initial beam conditions, magnet modeling, exact beam current, emittances, magnet locations, etc. Therefore, the implementation of the calculated element strengths in a real experiment does not yield true matching conditions. Under these circumstances, the "knobs" for the lens strengths must be adjusted on-line. The success of such a procedure relies at present completely upon the experience, judgement, and intuition of the operator. In a previous work [3], one of the authors showed the effectiveness of extremum seeking as non-model-based optimization technique to find a matching solution for a four-quadrupole-lens channel. Due to its non-model-based nature, extremum seeking is well suited to overcome the limitations described above for model-based optimization methods in terms of uncertainty handling. A hybrid scheme is now envisioned where the optimal lens strengths are computed off-line using extremum seeking or another optimization technique, and used as initial conditions ($\theta(0)$) for an on-line extremum seeking controller. Under this framework, the extremum seeking algorithm will be playing the role of a non-model-based adaptive controller, which is one of its unique characteristics, that ensures a well-matched beam at the end of the matching channel independently of uncertainties or changes in the system parameters.

The paper is organized as follows. In Section 2 the control problem is defined. Section 3 introduces the fundamentals of extremum seeking. The results of the simulation study are presented in Section 4. The paper is closed by a summary in Section 5.

II. Problem Definition

Assuming a continuous, elliptically-symmetric particle beam, we model its dynamics using the KV coupled-envelope equations [4]. Let the z coordinate represent the position along the design trajectory, and thus the xy plane is the transverse plane for the particle beam. At each z location, let $X(z)$ and $Y(z)$ represent the semi-axes of the

E. Schuster, N. Torres, and C. Xu are with the Department of Mechanical Engineering and Mechanics, Lehigh University, 19 Memorial Drive West, Bethlehem, PA 18015-0385, USA, schuster@lehigh.edu

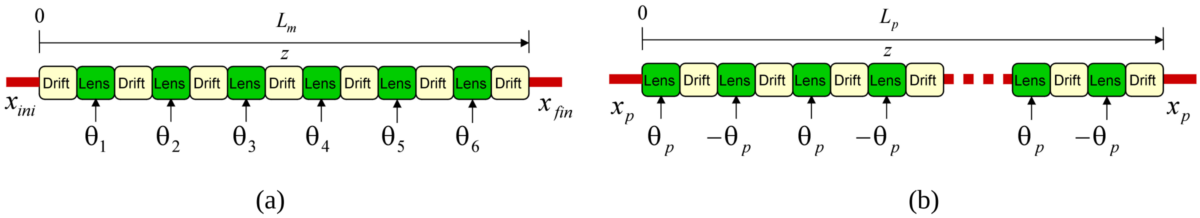


Fig. 1. Matching and periodic channels.

beam envelope in the x and y planes, respectively. The KV equations then appear as

$$X'' - \theta(z)X - \frac{2K}{X+Y} - \frac{\epsilon_X^2}{X^3} = 0 \quad (1)$$

$$Y'' + \theta(z)Y - \frac{2K}{X+Y} - \frac{\epsilon_Y^2}{Y^3} = 0 \quad (2)$$

where the prime indicates differentiation with respect to z, that varies from 0 to L (the combined length of both the matching and the periodic channels) and plays the role of "time". K is the beam perveance, ϵ_X and ϵ_Y are the effective emittances of the beam in the x and y planes, respectively. The focusing function $\theta(z)$, is shown in Figure 2-a, where κ is a constant, L_d is the drift length, and L_q is the quadropole lens length. The matching and periodic channel parameters (lens strengths) must satisfy the following constraints:

$$\begin{array}{c} |\theta_i| \leq 50 \text{ for } i=1,\ldots,6, \quad 0 \leq \theta_p \leq 50, \\ \theta_j \geq 0 \text{ for } j=1,3,5, \quad \theta_k \leq 0 \text{ for } k=2,4,6. \end{array} \quad (3)$$

We are given initial conditions for the beam at $z=0$, the transport system's entrance location. These conditions characterize the beam coming from the preceding section of the transport or accelerator system. They may be translated into initial conditions for the beam envelopes in the x plane (X_{ini}, X'_{ini}) and in the y plane (Y_{ini}, Y'_{ini}). In matching systems we are also given desired final conditions, or target conditions, at specific locations along the axis. For instance, at $z=L_m$, the exit location of the matching channel. We denote this target conditions as (X_{tar}, X'_{tar}) and (Y_{tar}, Y'_{tar}). They are prescribed by the acceptance requirements of the next section of the transport or accelerator system.

Denoting $x = [X \quad X' \quad Y \quad Y']^T$, we define

$$x_{ini} = x(0) = \begin{bmatrix} X_{ini} \\ X'_{ini} \\ Y_{ini} \\ Y'_{ini} \end{bmatrix}, \; x_{fin} = x(L_m) = \begin{bmatrix} X_{fin} \\ X'_{fin} \\ Y_{fin} \\ Y'_{fin} \end{bmatrix} \quad (4)$$

In addition, we define a target value for x denoted as $x_{tar} = [X_{tar} \quad X'_{tar} \quad Y_{tar} \quad Y'_{tar}]^T$. Fixed $\theta_p = \theta_p^m$, the acceptance requirement for the periodic channel, denoted x_p^m, is easily computed using numerical search algorithms. Defining, $x_{tar}|_{z=L_m} = x_p^m$, the matching solution θ_i^m, for $i=1,\ldots,6$, that makes $x_{fin} = x_{tar}|_{z=L_m} = x_p^m$, can be computed off-line using extremum seeking in a simulated environment or another model-based optimization technique. Defining $\theta = [\; \theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6 \quad \theta_p \;]^T$, Figure 2-b shows the envelopes X and Y for the matched beam as functions of z for

$$\theta^m = [\; 40 \quad -40 \quad 30 \quad -30 \quad 25 \quad -25 \quad 38 \;]^T \quad (5)$$

$$x_{ini} = \begin{bmatrix} 0.002887 \\ -0.01239 \\ 0.001053 \\ 0.01292 \end{bmatrix}, \; x_p^m = \begin{bmatrix} 0.001157 \\ -0.004266 \\ 0.001962 \\ 0.006098 \end{bmatrix}. \quad (6)$$

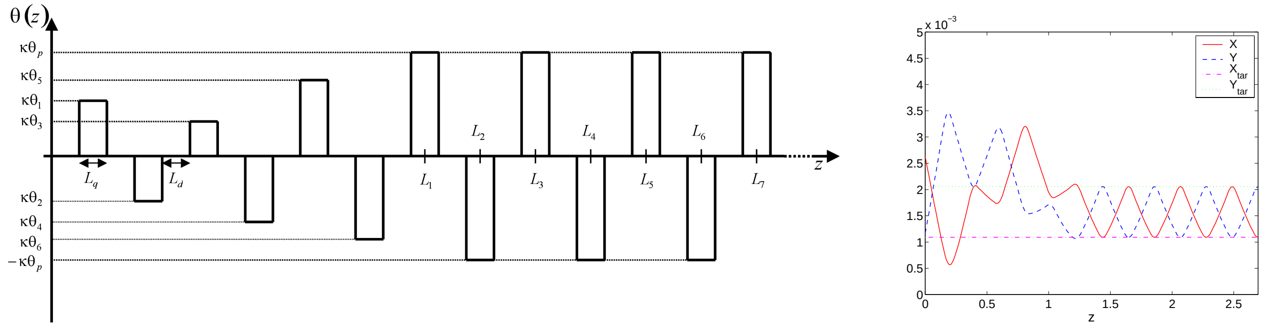
If θ is kept constant and equal to θ^m, Figures 3-a,b,c show how the matching properties are lost when: (1) there is an actuator fault (20% increase in the strength of the third quadrupole (θ_3), (2) there is a 50% increase for both the horizontal and vertical emittances, (3) there is a 10% change in the initial conditions X_{ini} and Y_{ini}.

We are interested in developing a controller that can successfully cope with these changes in operation conditions, preserving the matching properties of the system, by adaptively tuning the strengths of the lenses in the matching section (and eventually in the periodic section), and minimizing a functional that is a figure of the matching error. The problem is formulated as finite-"time" optimal adaptive control ($0 \leq z \leq L$), with bang-bang controls of fixed durations but varying intensities (i.e., with a very coarse discretization in "time" which results in a highly constrained waveform for the control θ as it is shown in Figure 2-a, for a plant that is nonlinear. This is far from being a standard optimization problem. To add complexity to the problem, we are seeking robustness against uncertainties of the system for a successful practical implementation of the control method.

III. Extremum Seeking

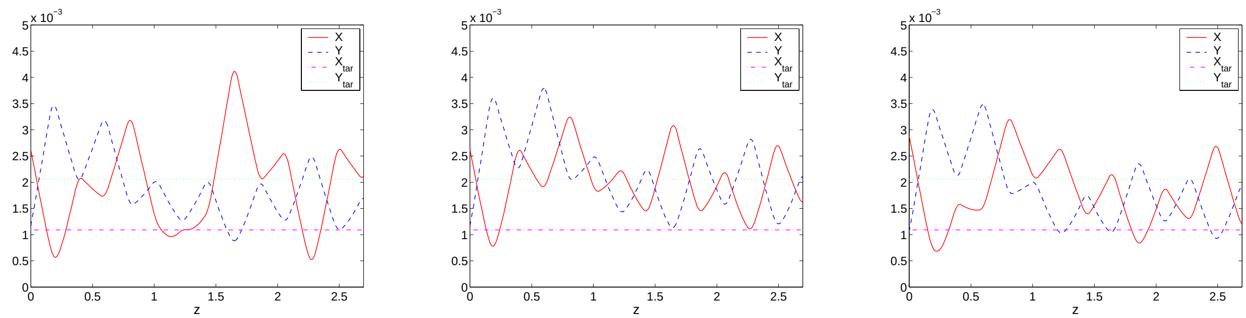
Extremum seeking control, a popular tool in control applications in the 1940-50's, has seen a resurgence in popularity as a real time optimization tool in different fields of engineering [5]. Extremum seeking is applicable in situations where there is a nonlinearity in the control problem, and the nonlinearity has a local minimum or a maximum. The parameter space can be multidimensional. In this paper we use extremum seeking for adaptive tuning of θ to preserve good matching, i.e., to minimize the following functional:

$$J = \{k_1 J_1 + k_2 J_2 + k_3 J_3\}^{\frac{1}{2}}, \quad (7)$$



(a) (b)

Fig. 2. Focusing function



(a) (b) (c)

Fig. 3. Beam envelopes evolution. (a) nominal, (b) actuator fault, (c) emittance change, (d) initial conditions change.

where k_1, k_2, k_3 are weight constants,

$$
\begin{aligned}
J_1 &= \sum_{i=1}^{7} M_i & (8)\\
M_i &= (X(L_i) - X_{tar})^2 + (Y(L_i) - Y_{tar})^2 \text{ for } i \text{ odd}\\
M_i &= (Y(L_i) - X_{tar})^2 + (X(L_i) - Y_{tar})^2 \text{ for } i \text{ even}\\
J_2 &= \sum_{i=2}^{7} N_i & (9)\\
N_i &= (X(L_i) - X(L_1))^2 + (Y(L_i) - Y(L_1))^2 \text{ for } i \text{ odd}\\
N_i &= (Y(L_i) - X(L_1))^2 + (X(L_i) - Y(L_1))^2 \text{ for } i \text{ even}\\
J_3 &= (X(L_1) - X_{tar})^2 + (Y(L_1) - Y_{tar})^2, & (10)
\end{aligned}
$$

and L_i, for $i = 1, \ldots, 7$, are specific locations along the periodic channel as it is shown in Figure 2-a. Defining J_1 as in (8) we are asking the controller to make the maximums and minimums of the periodic oscillation in the periodic channel equal to prespecified values X_{tar} and Y_{tar}. Defining J_2 as in (9) we are asking the controller to make all the envelop maximums equal, and all the envelope minimums equal, but without specifying values. In this way, when prespecified values X_{tar} and Y_{tar} are not achievable, at least we obtain symmetric periodic oscillations of period $2L_d + 2L_q$ in the periodic channel. The cost function component J_3 in (10) is usually used in combination with J_2 with appropriate weights to ask the controller to make the maximums and minimums of the periodic oscillation achieved by J_2 be as close as possible to prespecified values X_{tar} and Y_{tar}. We point out that, since X_{tar} and X_{tar} may be given arbitrarily, the dynamics of the transport channels are described by a system of nonlinear differential equations, and the lens input applied through θ is highly constrained in its waveform (Figure 2-a), there may not exist θ such that *perfect matching* is achieved ($J = 0$). Thus, we try to obtain the best possible *approximate* matching ($J =$ minimum).

We change θ after each beam "run." Thus, we employ the discrete time variant [6] of extremum seeking. The implementation is depicted in Figure 4, where q denotes the variable of the Z-transform. The high-pass filter is designed as $0 < h < 1$, and the modulation frequency ω is selected such that $\omega = \alpha\pi$, $0 < |\alpha| < 1$, and α is rational. The static nonlinear block $J(\theta)$ corresponds to one "run" of the system. The objective is to minimize J. If J has a global minimum its value is denoted by J^* and its argument by θ^*. Given X and Y at L_i, for $i = 1, \ldots, 7$, the output of the nonlinear static map, $J(k) = J(\theta(k))$, is then obtained by evaluating (7) and used to compute $\theta(k+1)$ according to the extremum seeking procedure in Figure 4, or written equivalently as

$$
\begin{aligned}
J_f(k) &= -hJ_f(k-1) + J(k) - J(k-1) & (11)\\
\xi(k) &= J_f(k) b \cos(\omega k - \phi) & (12)\\
\hat{\theta}(k+1) &= \hat{\theta}(k) - \gamma \xi(k) & (13)\\
\theta(k+1) &= \hat{\theta}(k+1) + a\cos(\omega(k+1)) . & (14)
\end{aligned}
$$

We are dealing with a multi-parameter extremum seeking procedure (6 or 7 parameters). Thus, we write

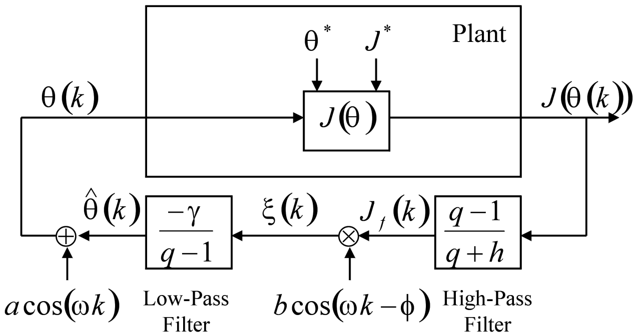


Fig. 4. Extremum seeking control scheme

$$\theta(k)=\begin{bmatrix}\theta_1(k)\\ \theta_2(k)\\ \theta_3(k)\\ \theta_4(k)\\ \theta_5(k)\\ \theta_6(k)\\ \theta_7(k)\end{bmatrix},\ \hat{\theta}(k)=\begin{bmatrix}\hat{\theta}_1(k)\\ \hat{\theta}_2(k)\\ \hat{\theta}_3(k)\\ \hat{\theta}_4(k)\\ \hat{\theta}_5(k)\\ \hat{\theta}_6(k)\\ \hat{\theta}_7(k)\end{bmatrix},\ \xi(k)=\begin{bmatrix}\xi_1(k)\\ \xi_2(k)\\ \xi_3(k)\\ \xi_4(k)\\ \xi_5(k)\\ \xi_6(k)\\ \xi_7(k)\end{bmatrix}.$$

The extremum seeking constants shown in Figure 4 are written as

$$\begin{aligned} a=b &= diag([\ a_1\quad a_2\quad a_3\quad a_4\quad a_5\quad a_6\quad a_7\ ]),\\ \gamma &= diag([\ \gamma_1\quad \gamma_2\quad \gamma_3\quad \gamma_4\quad \gamma_5\quad \gamma_6\quad \gamma_7\ ]). \end{aligned}$$

In addition, we denote

$$\cos(\omega k)=\begin{bmatrix}\cos(\omega_1 k)\\ \cos(\omega_2 k)\\ \cos(\omega_3 k)\\ \cos(\omega_4 k)\\ \cos(\omega_5 k)\\ \cos(\omega_6 k)\\ \cos(\omega_7 k)\end{bmatrix},\cos(\omega k-\phi)=\begin{bmatrix}\cos(\omega_1 k-\phi_1)\\ \cos(\omega_2 k-\phi_2)\\ \cos(\omega_3 k-\phi_3)\\ \cos(\omega_4 k-\phi_4)\\ \cos(\omega_5 k-\phi_5)\\ \cos(\omega_6 k-\phi_6)\\ \cos(\omega_7 k-\phi_7)\end{bmatrix}.$$

In a simulation environment, we understand by "run" the integration of the KV equations. In each iteration of the extremum seeking procedure, $\theta(k)$ is used to compute the focusing function $\theta(z)$, shown in Figure 2, which is in turn fed into the KV equations (1) and (2). Given x_{ini}, the KV equations are integrated to obtain $X(z)$, $Y(z)$, and finally X and Y at L_i, for $i=1,\ldots,7$, which are necessary to evaluate the cost function, $J(k)=J(\theta(k))$, in (7). In a real experiment, we understand by "run" one pulse of the accelerator, i.e., the passage of one beam bunch through the envelope detectors located at L_i, for $i=1,\ldots,7$. In this case, X and Y at L_i, for $i=1,\ldots,7$, are direct measures.

IV. Simulation Results

The physical parameters used in the simulations presented in this section are $K=2.7932\times10^{-6}$, $\epsilon_X=6\times10^{-6}$, $\epsilon_Y=6\times10^{-6}$, $\kappa=2.6689$, $L_d=0.1488$, $L_q=0.0610$, and $L=13L_d+12.5L_q=L_7$ (as shown in Figure 2). In addition, the extremum seeking parameters are $h=0.4$, $\omega_i=\omega_{base}^i\times\pi$, for $i=1,\ldots,6$ or 7, where $\omega_{base}=0.95$, $\gamma_i=0.1\frac{M(\omega_1)}{M(\omega_i)}$, and $\phi_i=-\phi(\omega_i)$, for $i=1,\ldots,6$ or 7, where $M(\omega)$ and $\phi(\omega)$ are respectively the magnitude and phase of the frequency response of the extremum-seeking high-pass filter shown in Figure 4. For all the simulations, the nominal initial condition, x_{ini}^n, of the beam at the entrance of the channel is that given in (6). In addition, the nominal values of the quadrupole strengths, θ^n, are equal to those given in (5). The initial conditions for the extremum seeking parameters in all the simulations are equal to the nominal values, i.e., $\theta(0)=\theta^n$. The target values involved in the computation of the cost function J in (7) are $X_{tar}=0.001092$ and $Y_{tar}=0.002055$. Following we study the performance of the extremum-seeking adaptive controller to regulate the system around the nominal profile shown in Figure 2-b in the presence of faults or changes.

a) Actuator Fault: In this case we study the response of our controller to a drift in one of the quadrupoles of the matching channel. We rewrite (14) as

$$\theta(k+1)=\frac{k}{D}d+\hat{\theta}(k+1)+a\cos(\omega(k+1)) \qquad (15)$$

where D is the drift rate and the column vector d is used to correlate such drift with a specific quadrupole. In this simulation study, presented in Figure 5, $D=10^{-2}$ and $d=[\ 0\quad 0\quad 1\quad 0\quad 0\quad 0\quad 0\ ]^T$, indicating that the drift is present in the third quadrupole. In addition, we assume $\theta_p=38=$ cte, i.e., we use the extremum-seeking controller to tune only the strengths of the six quadrupoles in the matching channel. The controller successfully sustain the nominal beam profile shown in Figure 2-b in spite of the actuator drift. Figure 5-a shows that the cost function (7), which was defined taking $k_1=1$, $k_2=0$ and $k_3=0$, is kept at its minimum run after run. This is possible due to the adaptive tuning of the quadrupole strengths θ_i, for $i=1,\ldots,6$. Figure 5-b shows how the outputs of the extremum seeking controller, specially the output associated with the third quadrupole ($\hat{\theta}_3$), are varied to compensate the drift, and therefore to keep θ in Figure 5-c close to its nominal value.

b) Initial Conditions Change: In this case we study the response of our controller to a change in the geometrical characteristics of the beam at the entrance of the matching section produced at the 100th run ($X_{ini}=1.1X_{ini}^n$ and $Y_{ini}=0.9Y_{ini}^n$). This change in initial condition may be produced by a fault in the preceding section of the accelerator. In this simulation study, presented in Figure 6, we again assume $\theta_p=38=$ cte, i.e., we use the extremum-seeking controller to tune only the strengths of the six quadrupoles in the matching channel. The cost function (7), whose evolution is shown in Figure 6-a, is also defined here taking $k_1=1$, $k_2=0$ and $k_3=0$. Figure 6-a shows through the sudden increase in the value of the cost function at the 100th run how the matching properties of the system are transitorily lost. This can be also noted by comparing Figure 6-b and Figure 6-c, showing the beam profile before and after the change in initial conditions. The controller successfully recovers the matching properties after the transient, as it is shown in Figure 6-e, by adaptively tuning the strengths of the quadrupoles in the matching channel (Figure 6-d). Figure 6-f shows the matched beam

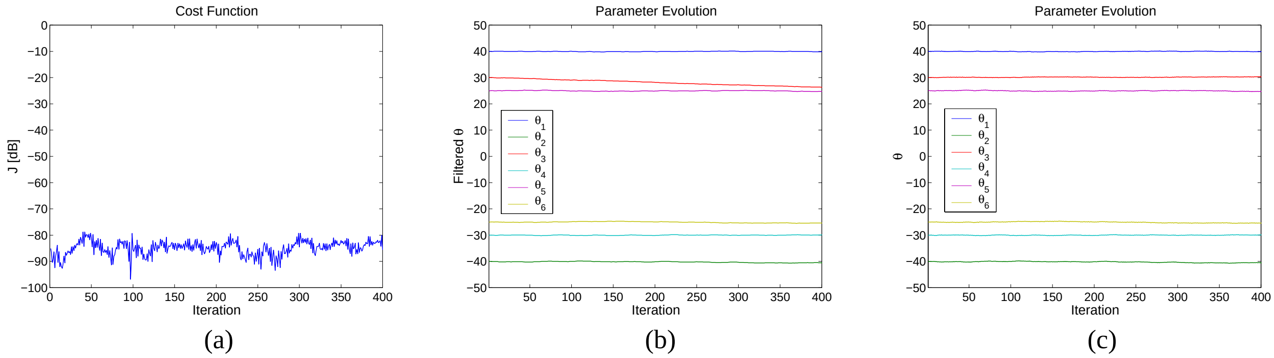


Fig. 5. Actuator drift in the third quadrupole of the matching section.

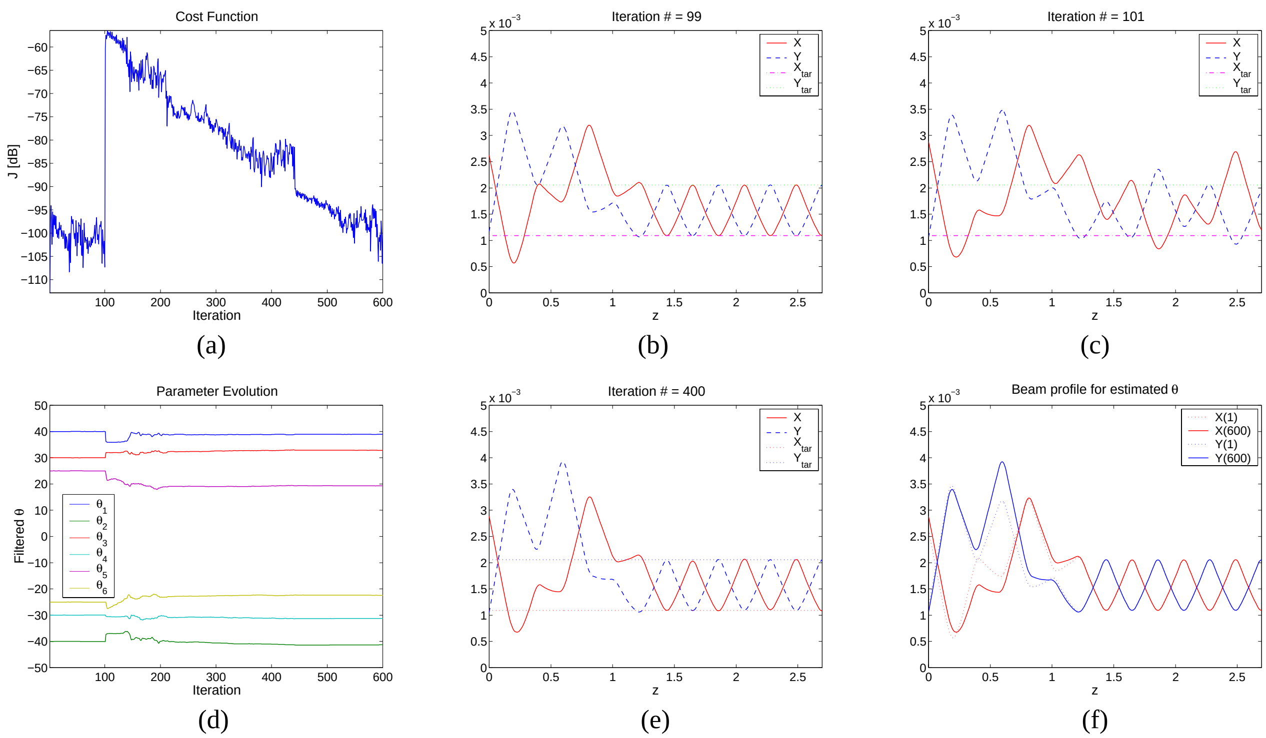


Fig. 6. Change of geometrical characteristics of the beam at the entrance of the matching channel at the 100th iteration.

profiles before and after the change in initial conditions. It is possible to note how the profile within the matching section is changed by the controller in order to preserve the characteristics of the beam profile in the periodic section.

c) Multiple Faults or Changes: In this case we consider simultaneous fault or change occurrences. At the 100th run, a 10$\%$ change in the third quadrupole of the matching section, a 10 $\%$ change in the geometrical characteristics of the beam at the entrance of the matching section ($X_{ini} = 1.1X_{ini}^n$ and $Y_{ini} = 0.9Y_{ini}^n$), and a 50$\%$ change in the emittance of the beam produced ($\epsilon_X = 9 \times 10^{-6}$, $\epsilon_Y = 9 \times 10^{-6}$) are simulated. The cost function (7), whose evolution is shown in Figure 7-a, is now defined taking $k_1 = 0$, $k_2 = 1$ and $k_3 = 0.1$. This selection ($k_1 = 0$, $k_2 \neq 0$) is motivated by the fact that given the properties of the beam (emittance and perveance), and fixed the initial conditions of the beam x_{ini}, it is not always possible to achieve arbitrary values of X_{tar} and Y_{tar} (desired minimum and maximum of the symmetric beam profile in the periodic channel). Under these circumstances, the symmetry of the beam is prioritized over its geometrical measures, as it is manifested in the definition of the cost function J. Basically, we give to the system one more degree of freedom to accommodate the change in the properties of the beam (emittance). In this simulation study, presented in Figure 7, we do not assume $\theta_p = $ cte, i.e., we use the extremum-seeking controller to tune not only the strengths of the six quadrupoles in the matching channel but also the common strength of the quadrupoles in the matching section. The goal is to use the extra parameter $\theta_7 = \theta_p$ in the extremum seeking controller to make the size of the beam as close as possible to that defined by X_{tar} and Y_{tar} ($k_3 \neq 0$). Figure 7-a shows through the sudden increase in the value of the cost function at the 100th run how the matching properties of the system are lost transitorily. This can be

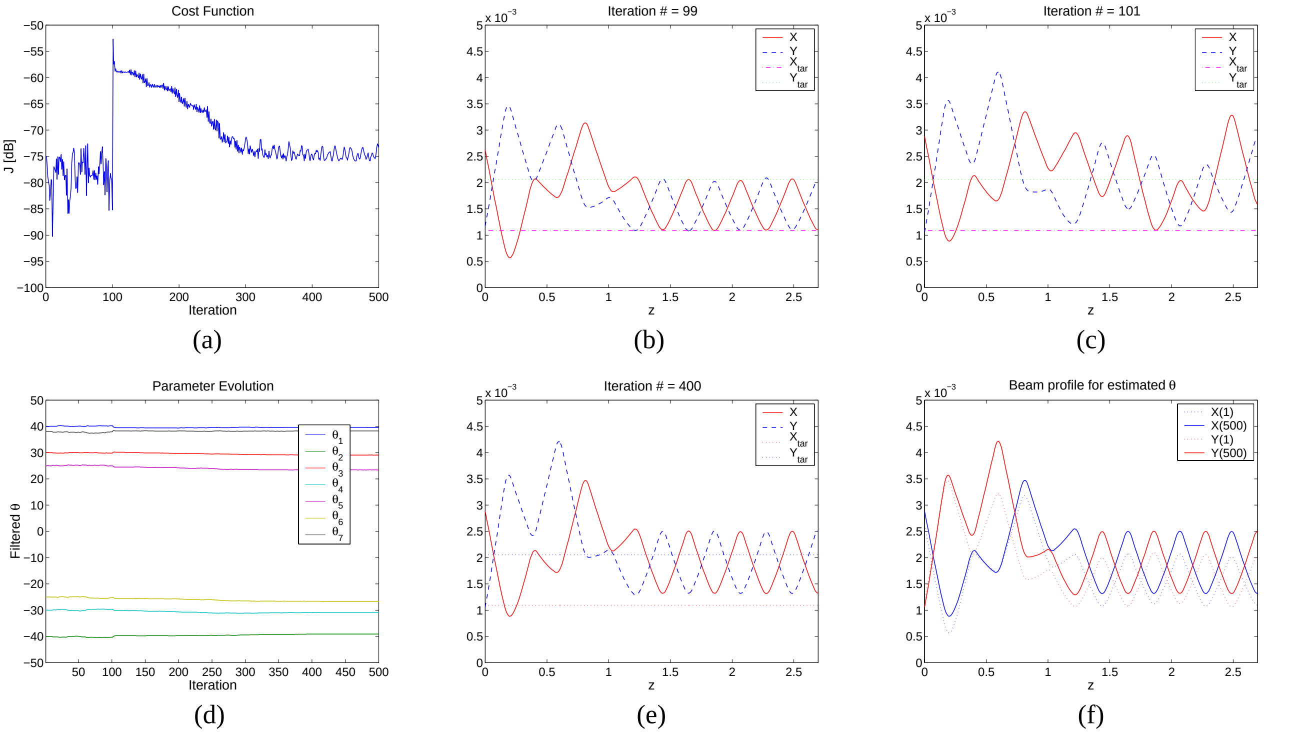


Fig. 7. Multiple faults or changes at the 100th iteration: actuator step in the third quadrupole, change in the initial conditions and emittance of the beam.

also noted by comparing Figure 7-b and Figure 7-c, showing the beam profile before and after the multiple changes. In addition, Figure 7-a shows how the system is driven from a minimum before the changes at the 100th run to another different minimum after a transient that follows the changes. By adaptively tuning the strengths of the quadrupoles in both the matching and the periodic channels, as shown in Figure 7-d, the controller successfully recovers the matching properties after the transient (Figure 7-e). However, in this case only the "shape", and not the "value", of the beam profile can be preserved within the periodic section. We must emphasize that this is not a constraint of the controller but of the system itself. Figure 7-f shows the matched beam profiles before and after the multiple changes. In this case, not only the profile within the matching section must be changed by the controller, but also the size of the beam within the periodic section, in order to guarantee symmetry of the beam (maximums of X and Y are identical, minimums of X and Y are identical).

V. Conclusions

A multi-parameter, extremum-seeking, non-model-based, adaptive controller has been designed, and successfully tested in simulations, for the tuning of the lens strengths in a 6-lens matching channel combined with a periodic channel. Based on the promising results obtained in the simulation study, it is anticipated that the scheme can play an important role in real-time adaptive control of beam envelopes in particle accelerators. Due to its non-model-based nature, which represents an advantage with respect to other model-based optimization techniques, the extremum-seeking controller can cope with model uncertainties and system errors, faults, or changes.

Another unique property of this type of controller is the flexibility that the designer has to define control goals by the appropriate definition of the cost function J. Due to this flexibility, constraints of the system as well as competing objectives can be introduced into the controller. Present work by the authors include the development of analytical expressions for sensitivity of the matched beam (against "errors" in the actuators (quadrupoles)) that can be incorporated into the extremum-seeking adaptive controller functional in order not only to converge to a matching solution but also to converge to the least sensitive one (if the degrees of freedom allow it).

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