Robust Control of the Poloidal Magnetic Flux Profile in the Presence of Unmodeled Dynamics

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***Abstract*—The potential operation of a tokamak fusion reactor in a highly-efficient, steady-state mode is directly related to the achievement of certain types of radial profiles for the current flowing toroidally in the device. The evolution in time of the toroidal current profile in tokamaks is related to the evolution of the poloidal magnetic flux profile, which is modeled in normalized cylindrical coordinates using a nonlinear partial differential equation (PDE) usually referred to as the magnetic diffusion equation. We propose a robust control scheme to regulate the poloidal magnetic flux profile in tokamaks in the presence of model uncertainties. These uncertainties come mainly from the resistivity term of the magnetic diffusion equation. First we either simulate the magnetic diffusion equation or carry out experiments to generate data ensembles, from which we then extract the most energetic modes to obtain a reduced order model based on proper orthogonal decomposition (POD) and Galerkin projection. The obtained reduced-order model corresponds to a linear state space representation with uncertainty. Taking advantage of the structure of the state matrices, the reduced order model is reformulated into a robust control framework, with the resistivity term as an uncertain parameter. An H_∞ controller is designed to minimize the regulation/tracking error. Finally, the synthesized model-based robust controller is tested in simulations.**

I. Introduction

Setting up a suitable current profile, which is proportional to the spatial derivative of the poloidal flux profile, has been demonstrated to be a key condition for one possible advanced scenario with improved confinement and possible steady-state operation [1]. One approach to current profile control is to focus on creating the desired current profile during the plasma current ramp-up and early flat-top phases (finite-time optimal control problem) with the aim of maintaining this target profile during the subsequent phases of the discharge (regulation problem).

Our previous work includes the investigation of the use of extremum seeking [2] and nonlinear programming [3] to achieve open-loop solutions for the optimal control problem defined during the ramp-up and early flat-top phases. The time evolutions of the control inputs are obtained in the interval $[0,T]$ in order to minimize the quadratic error between actual and desired current profiles at time T. The work is aimed at saving long trial-and-error periods of time currently spent by fusion experimentalists trying to manually adjust the time evolutions of the actuators to achieve the desired current profile at some time T within a prespecified window $[T_1,T_2]$.

These open-loop solutions depend on the plasma resistivity, and therefore on the electron temperature, whose dynamics is very difficult to be predicted by simple control-oriented models [4]. In this paper, we take into account the un-modeled temperature dynamics by considering the resistivity coefficient in the magnetic diffusion equation as an uncertainty. After reducing the dimensionality of the magnetic diffusion equation by combining proper orthogonal decomposition (POD) and Galerkin projection, the model for the poloidal flux is written within a robust control framework. A robust controller minimizing the H_∞ norm of the sensitivity function of the closed loop system is then designed to reduce the tracking/regulation error.

The paper is organized as follows. The dynamic model for the poloidal flux is introduced in Section II. The model reduction technique is explained in Section III. Section IV describes how the model is written within a robust control framework. The controller is designed and tested in simulations in Section V. The paper is closed with the conclusions in Section VI.

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II. Current Profile Evolution Model

The evolution of the poloidal flux in normalized cylindrical coordinates is given by the magnetic diffusion equation,

$$\frac{\partial \psi}{\partial t}=\nu(\hat{\rho})\frac{1}{\hat{\rho}}\frac{\partial}{\partial \hat{\rho}}\left(\hat{\rho} f_4(\hat{\rho})\frac{\partial \psi}{\partial \hat{\rho}}\right)+f_2(\hat{\rho})u_1(t) \qquad (1)$$

with boundary conditions

$$\left.\frac{\partial \psi}{\partial \hat{\rho}}\right|_{\hat{\rho}=0}=0, \quad \left.\frac{\partial \psi}{\partial \hat{\rho}}\right|_{\hat{\rho}=1}=u_2(t), \qquad (2)$$

initial condition

$$\psi_0(\hat{\rho})=\psi(\hat{\rho},0), \qquad (3)$$

and

$$\nu(\hat{\rho},t) \quad = \quad K_\eta(\hat{\rho})\eta(T_e), \qquad (4)$$

$$u_1(t) = \frac{\sqrt{P_{tot}(t)}}{I(t)}, \quad (5)$$

$$u_2(t) = K_I I(t), \quad (6)$$

where t is the time, $\hat{\rho} = \frac{\rho}{\rho_b}$ is the normalized version of an arbitrary coordinate ρ indexing the magnetic surfaces (ρ_b denotes its value at the boundary), ψ is the poloidal magnetic flux, η is the plasma resistivity, T_e is the plasma electron temperature, $P_{tot}(t)$ is the total power of the non-inductive current source, $I(t)$ is the plasma current, $f_2(\hat{\rho})$, $f_4(\hat{\rho})$, and $K_\eta(\hat{\rho})$ are spatial functions, and K_I is a constant (see [5] for a detailed model description).

In practice it is very difficult to accurately predict the time evolution of the electron temperature T_e, and consequently of the plasma resistivity $\eta(T_e)$, by a model that is simple enough for control design. Therefore, in this work we integrate $\eta(T_e)$ into $v(\hat{\rho},t)$ and model it as an uncertainty as explained below.

III. Model Reduction Using POD/Galerkin

We first solve the parabolic PDE system on the grid $\mathcal{Q}_{ij} = (\hat{\rho}_i, t_j)$, where i, j are integers with $1 \le i \le m; 1 \le j \le n$. The set $\mathscr{V} = \text{span}\{\psi_1, \cdots, \psi_n\} \subset \mathbb{R}^m$ refers to a data ensemble consisting of the snapshots $\{\psi_j\}_{j=1}^n$ obtained from the simulation. The goal of the POD method is to find an orthonormal basis $\{\phi_k\}_{k=1}^l$ such that for some predefined $1 \le l \le d$, where $d = \dim \mathscr{V} \le m$, the reconstruction error for the snapshots is minimized, i.e.,

$$\min_{\{\phi_k\}_{k=1}^l} \frac{1}{n} \sum_{j=1}^n \left\| \psi_j - \sum_{k=1}^l (\psi_j, \phi_k) \phi_k \right\|^2, \quad (7)$$

subject to

$$(\phi_i, \phi_j) = \delta_{ij}, \quad 1 \le i \le l, \quad 1 \le j \le i,$$

where $\|\psi\| = \sqrt{\psi^T \psi}$ and $(\cdot,\cdot)$ denotes the inner product in the space $L^2([0,1])$.

Let $\Lambda_1 > \ldots > \Lambda_l > \ldots > \Lambda_d > 0$ denote the positive eigenvalues of the correlation matrix K, defined as $K_{ij} = \frac{1}{n}(\psi_j, \psi_i)$, for $i, j = 1, \ldots, n$, and $v_1, \ldots, v_l, \ldots, v_d$ the associated eigenvectors, where $d = \text{rank}(K)$. Then, the POD basis functions take the form [6]

$$\phi_k = \frac{1}{\sqrt{\Lambda_k}} \sum_{j=1}^n (v_k)_j \psi_j = \frac{1}{\sqrt{\Lambda_k}} Y v_k, \, (k = 1, \ldots, d), \quad (8)$$

where $(v_k)_j$ is the j-th component of the eigenvector v_k and $Y = (\psi_1, \cdots, \psi_n)$ is the collection of all the snapshots. Moreover, the error (energy ratio) associated with the approximation with the first l POD modes is

$$\varepsilon_l = \frac{1}{n} \sum_{j=1}^n \left\| \psi_j - \sum_{k=1}^l (\psi_j^T \phi_k) \phi_k \right\|^2 = \sum_{k=l+1}^d \Lambda_k. \quad (9)$$

Let $V = \left\{ z | z, \frac{dz}{dx} \in L^2(\hat{\rho}) \right\}$, and $\phi(\hat{\rho}) \in V$ be a test function, where $\hat{\rho} \in [0,1]$. Let $V_{POD} = span\{\phi_1, \phi_2, \phi_3, \phi_4, ..., \phi_l\} \subset V$ be a space spanned by the POD modes obtained from the model reduction process for $\psi(\hat{\rho},t)$. Let

$$\psi(\hat{\rho},t) \approx \psi^l(\hat{\rho},t) = \sum_{k=1}^l \beta_k(t) \phi_k(\hat{\rho}), \quad (10)$$

where $\phi_k(\hat{\rho}) \in V_{POD}$, $k = 1, 2, ..., l$. Similarly, let $W_{POD} = span\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, ..., \varphi_n\} \subset W$ be a space spanned by the POD modes obtained from the model reduction process for $v(\hat{\rho},t)$. We write

$$v(\hat{\rho},t) \approx v^n(\hat{\rho},t) = \sum_{i=1}^n \gamma_i \varphi_i(\hat{\rho}), \quad (11)$$

where $\Gamma = (\gamma_1, ..., \gamma_n)^T \in \mathbb{R}^n$ is the uncertainty vector, and $\varphi_i(\hat{\rho}) \in W_{POD}$, $i = 1, 2, ..., n$. The vector Γ is the finite dimensional approximation of $v(\hat{\rho},t)$ with respect to the obtained POD modes. Each element γ_i of Γ is a time-varying uncertainty associated with $\varphi_i(\hat{\rho})$, and $\gamma_i = \gamma_i^0(1 + \delta_i)$ with $|\delta_i| < 1$, for all i.

We multiply both sides of equation (1) by $\hat{\rho}\phi(\hat{\rho})$, with $\phi(\hat{\rho}) \in V$, and integrate by parts over the spatial domain $[0,1]$ taking into account (10)–(11) and using the notation

$$< g_1, g_2, ... g_n > \triangleq \int_0^1 g_1(\hat{\rho}) g_2(\hat{\rho}) ... g_n(\hat{\rho}) \hat{\rho} d\hat{\rho}, \quad (12)$$

and

$$\begin{aligned} M_{jk} &= < \phi_k, \phi_j >, \\ K_{jk} &= \sum_{i=1}^n \gamma_i (< f_4 \phi_k', \phi_j', \varphi_i > + < f_4 \phi_k', \phi_j, \varphi_i' >) \\ P_j &= < \phi_j, f_2 >, \quad Q_j = \sum_{i=1}^n \gamma_i f_4(1) k_3 \phi_j(1) \varphi_i(1), \end{aligned} \quad (13)$$

where $F' = \frac{\partial F}{\partial \hat{\rho}}$, to obtain a matrix representation for the reduced order model

$$M \frac{dx}{dt} = -Kx + P u_1(t) + Q u_2(t), \quad (14)$$

where $x(t) = (\beta_1, ..., \beta_l)^T \in \mathbb{R}^l$, $M, K \in \mathbb{R}^{l \times l}$, $P, Q \in \mathbb{R}^l$. The vector $x(t)$ is the finite dimensional approximation of $\psi(\hat{\rho},t)$ with respect to the obtained POD modes. The components of the initial state are given by

$$x^i(t_0) = x_0^i = (\psi(t_0), \phi_i), \quad i = 1, \ldots, l, \quad (15)$$

where $x_0 \in \mathbb{R}^{l \times 1}$ and ϕ_i, for $i = 1, \ldots, l$, are POD modes.

Assuming that all the states are measurable, the state-space representation of the reduced-order model is given by

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (16)$$

where $A = -M^{-1}K \in \mathbb{R}^{l \times l}$, $B = M^{-1}[P \; Q] \in \mathbb{R}^{l \times 2}$, $C = I_l$ is a $l \times l$ identity matrix, $D = 0$ and $u(t) = [u_1(t) \; u_2(t)]^T$.

IV. Model in Robust Control Framework

A system with state space representation A, B, C, D has a transfer function $G(s) = D + C(sI_l - A^{-1})B$, where l is the number of states in the system. We can write the transfer function as a linear fractional transformation

$$\begin{aligned} G(s) &= F_u(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \frac{1}{s}I_l) \qquad (17) \\ &= D + C\frac{1}{s}I_l(I_l - A\frac{1}{s}I_l)^{-1}B = D + C(sI_l - A^{-1})B. \end{aligned}$$

To make the uncertainty in the state-space system (16) explicit, the matrices K and Q can be rewritten as

$$K = K_0 + \sum_{i=1}^{n} \delta_i K_i, \quad Q = Q_0 + \sum_{i=1}^{n} \delta_i Q_i. \qquad (18)$$

where

$$\begin{aligned} K^0_{jk} &= \sum_{i=1}^{n} \gamma_i^0 (< f_4\phi'_k, \phi'_j, \varphi_i > + < f_4\phi'_k, \phi_j, \varphi'_i >), \qquad (19) \\ K^i_{jk} &= \gamma_i^0 (< f_4\phi'_k, \phi'_j, \varphi_i > + < f_4\phi'_k, \phi_j, \varphi'_i >), \qquad (20) \\ Q^0_j &= \sum_{i=1}^{n} \gamma_i^0 f_4(1) k_3 \phi_j(1) \varphi_i(1), \qquad (21) \\ Q^i_j &= \gamma_i^0 f_4(1) k_3 \phi_j(1) \varphi_i(1). \qquad (22) \end{aligned}$$

Then, we define the matrix M_a as a general affine state-space uncertainty

$$M_a = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_0 + \sum_{i=1}^{n} \delta_i A_i & B_0 + \sum_{i=1}^{n} \delta_i B_i \\ C_0 + \sum_{i=1}^{n} \delta_i C_i & D_0 + \sum_{i=1}^{n} \delta_i D_i \end{bmatrix} \qquad (23)$$

with $A_0 = -M^{-1}K_0 \in \mathbb{R}^{l\times l}$, $A_i = -M^{-1}K_i \in \mathbb{R}^{l\times l}$ $B_0 = M^{-1}[P\ Q_0] \in \mathbb{R}^{l\times 2}$, $B_i = M^{-1}[0\ Q_i]$, $C_0 = I_l$, $C_i = 0$ and $D_0 = D_i = 0$ for all $i = 1, 2, ..., n$.
This uncertainty can be formulated into a linear fractional transformation by achieving the smallest number of repeated blocks using the method outlined in [7]. With this purpose, the matrix J_i is formed as

$$J_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \in \mathbb{R}^{2l\times(l+2)}. \qquad (24)$$

Using singular value decomposition and grouping terms, the matrix J_i can be expressed as

$$J_i = U_i \Sigma_i V_i^* = (U_i\sqrt{\Sigma})(\sqrt{\Sigma}V_i^*) = \begin{bmatrix} L_i \\ W_i \end{bmatrix} \begin{bmatrix} R_i \\ Z_i \end{bmatrix}^*, \qquad (25)$$

where A^* denotes the complex conjugate transpose of A. Then, the uncertainty can be written as

$$\delta_i J_i = \begin{bmatrix} L_i \\ W_i \end{bmatrix} [\delta_i I_{q_i}] \begin{bmatrix} R_i \\ Z_i \end{bmatrix}^*, \qquad (26)$$

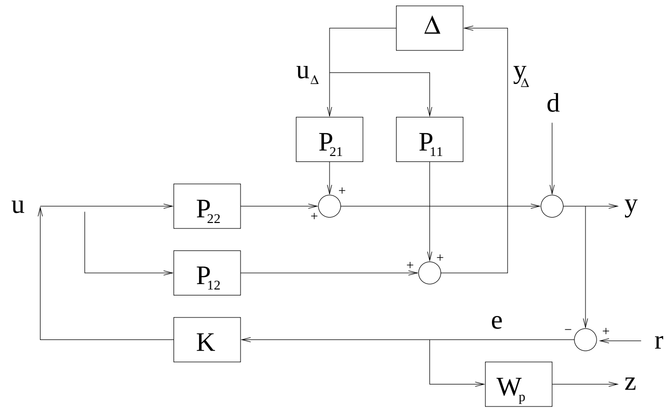

Fig. 1: Overall feedback system.

where $q_i = 1$ for all $i = 1, 2, ..., n$ in this case. Therefore, the matrix M_a can be written as

$$M_a = M_{11} + M_{12}\Delta M_{21}, \qquad (27)$$

where

$$M_{11} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \qquad M_{12} = \begin{bmatrix} L_1 & \cdots & L_n \\ W_1 & \cdots & W_n \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} R_1^* & Z_1^* \\ \vdots & \vdots \\ R_n^* & Z_n^* \end{bmatrix} \qquad \Delta = \begin{bmatrix} \delta_1 I_{q_1} & & 0 \\ & \ddots & \\ 0 & & \delta_n I_{q_n} \end{bmatrix}.$$

This is equal to the lower linear fractional transformation

$$M_a = F_l(M, \Delta) \qquad (28)$$

with

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & 0, \end{bmatrix}.$$

Finally, the transfer function $G(s)$ of the uncertain state-space model is written as

$$\begin{aligned} G(s) =& F_u(F_l(M, \triangle), \frac{1}{s}I_l) = F_l(F_u(M, \frac{1}{s}I_l), \triangle) \\ =& F_l(P', \triangle) = F_u(P, \triangle). \end{aligned} \qquad (29)$$

The goal is to design a controller that can robustly track the optimal open-loop trajectories of magnetic flux ψ and meet special performance requirements. Therefore, let us consider the reference r and disturbance d as inputs, and a weighted version of the tracking error as the output $z = W_p e$, where W_p is a weight chosen by the designer. The overall feedback system is shown in Fig. 1. Then, the generalized plan P^* from $[u_\Delta\ d\ r\ u]^T \in \mathbb{R}^{13\times 1}$ to $[y_\Delta\ z\ e]^T \in \mathbb{R}^{12\times 1}$ is (see Fig. 2)

$$P^* = \begin{bmatrix} P_{11} & 0 & 0 & P_{12} \\ -W_p P_{21} & -W_p & W_p & -W_p P_{22} \\ -P_{21} & -1 & 1 & P_{22} \end{bmatrix}.$$

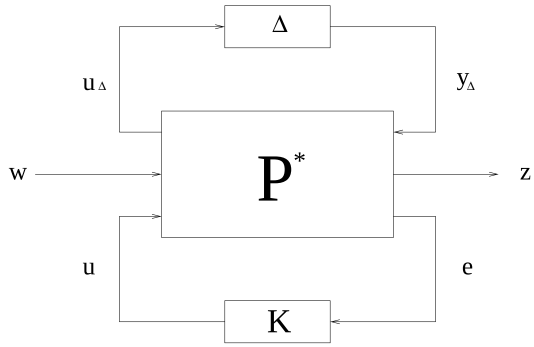

Fig. 2: Robust control framework for augmented plant P^*.

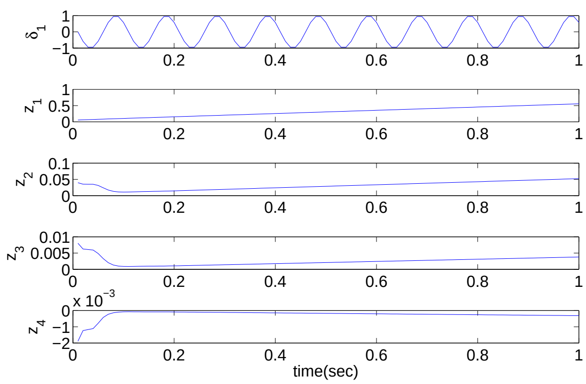

Fig. 3: System response without any control.

V. Controller Synthesis and Simulations

The control objective is to track the optimal open-loop control references r. The optimization problem is to find an H_∞ controller K to minimize the cost function

$$\|W_p S\|_\infty, \tag{30}$$

where $S = (1+KG)^{-1}$ and the weight unction W_p is defined as

$$W_p = \frac{(s/M_p^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{*1/2})^2}, \tag{31}$$

with $M = 1$, $A^* = 10^{-4}$ and $\omega_B^* = 10^6$ [8]. The uncertain parameter δ_i ranges from -1 to 1. This is the range of values for which the system should be stabilized so that the robust controller can be considered a suitable design.

For the simulation study presented in this section we assume that that there is no external disturbance, i.e. $d = 0$. The reference signals r are step functions and the time-varying uncertain parameters δ_i's are sinusoidal functions of 20πHz. The frequency of the sinusoidal functions affects the transient response (rise time, settling time and overshoot) but does no effect stability. Only four POD modes are used for model reduction. Figures 3 and 4 compare the weighted tracking errors for the uncontrolled and controlled (H_∞ controller K) systems respectively.

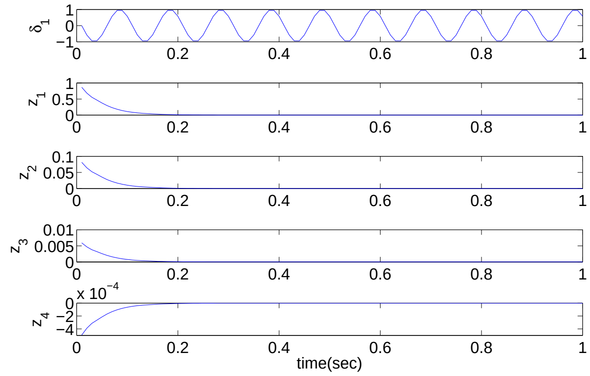

Fig. 4: System response with the H_∞ controller K.

VI. Conclusions

In this paper, we consider a control-oriented dynamic model describing the evolution of the poloidal flux during the inductive phase of the tokamak discharge. Using the POD/Garlekin technique, we reformulate this PDE model into a low dimensional ODE model that preserves the dominant dynamics of the original parabolic PDE. The resistivity term is modeled as an uncertainty and the model is rewritten within a robust control framework $\Delta - P^* - K$. A robust controller is synthesized to minimize the H_∞ norm of a weighted version of the sensitivity transfer function relating z and r, and therefore to minimize the weighted tracking error. The simulation study shows that the proposed robust controller stabilizes the system and improves the tracking performance when compared to the uncontrolled case.

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