On the Stability of the Kapchinskij-Vladimirskij Equation*

Chao Xu, Eugenio Schuster and Christopher K. Allen

Abstract

We discuss the stability of the linearized Kapchinskij-Vladimirskij (KV) equation around a matched solution, which is a linear periodic Hamiltonian system. By using the averaging method and asymptotic analysis, we find stability boundaries for the linearized beam envelope system without integrating the KV equation numerically.

INTRODUCTION

The transport of a charged particle beam in an alternating-gradient focusing magnetic field has a wide spectrum of applications, ranging from scientific research to industrial processes. Stability of the envelope profile of the particle beam is an important problem in particle transport (e.g., [1]). In this paper we study the linear stability of the Kapchinskij-Vladimirskij (KV) equations [1],

$$a'' + \kappa(s)a - 2K(a+b)^{-1} - \varepsilon_x^2 a^{-3} = 0, \quad (1)$$

$$b'' - \kappa(s)b - 2K(a+b)^{-1} - \varepsilon_y^2 b^{-3} = 0, \quad (2)$$

where a and b represent the semi-axes of the elliptical beam envelope in the transverse plane. The variable s represents the axial displacement along the beam propagation direction. We assume an alternating-gradient quadrupole focusing lattice through which the ion beam propagates and is transported. The periodicity of the quadrupole focusing lattice function $\kappa(s)$ is S. The normalized beam emittances are denoted as ε_x and ε_y, and the self-field perveance as K. When $a = b = r$, the KV equations degenerate to the 1D case ($\varepsilon_x = \varepsilon_y = \varepsilon$)

$$r'' + \kappa(s)r - Kr^{-1} - \varepsilon^2 r^{-3} = 0. \quad (3)$$

LINEARIZATION

We denote the periodic matched solutions by X_0 and Y_0, which satisfy

$$X_0'' + \kappa X_0 - 2K(X_0+Y_0)^{-1} - \varepsilon_x^2 X_0^{-3} = 0, \quad (4)$$

$$Y_0'' - \kappa Y_0 - 2K(X_0+Y_0)^{-1} - \varepsilon_y^2 Y_0^{-3} = 0. \quad (5)$$

We write $a(s) = X_0(s) + x(s)$ and $b(s) = Y_0(s) + y(s)$, where $x(s)$ and $y(s)$ represent small deviations from the matched solution. By substituting $a(s)$ and $b(s)$ into (1)-(2), we can obtain the linear non-autonomous system

$$x'' + a_1 x + a_0 y = 0, \quad y'' + a_0 x + a_2 y = 0, \quad (6)$$

where $a_0 = 2K(X_0+Y_0)^{-2}$, $a_1 = \kappa + 3\varepsilon_x^2 X_0^{-4} + a_0$ and $a_2 = -\kappa + 3\varepsilon_y^2 Y_0^{-4} + a_0$. The linearized version of the 1D KV equation (3) can be obtained by the same procedure.

We let $\zeta(s) = [x(s), y(s), x'(s), y'(s)]^T$, and rewrite (6) in the following matrix form

$$\zeta' = \frac{d\zeta}{ds} = JH(s)\zeta(s), \quad (7)$$

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, H = \begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_0 & a_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The state transition matrix $\Phi(s, s_0)$ of the linear system (7) solves $\Phi'(s, s_0) = JH(s)\Phi(s, s_0)$, $\Phi(s_0, s_0) = I$.

STABILITY BY AVERAGING

In this section, we use some stability results of Hamiltonian systems (e.g., [2]) to obtain stability conditions for the linearized KV equation without computing the transition matrix. The stability conditions are expressed in terms of the averaged system.

Definition 1 *Given the non-autonomous linear system (7), we define the associated averaged system as* $\zeta' = JH_{\text{av}}\zeta$, $\zeta(0) = \zeta_0$, *where* $JH_{\text{av}} = \frac{1}{S}\int_0^S JH(s)ds$.

Definition 2 *A matrix* $H(s)$, $(0 \le s \le S)$ *of degree* $2n$ *is said to be of positive type if the following two conditions are satisfied:* a. *for each* $s \in [0, S]$, *the corresponding Hermitian form is nonnegative, i.e.,* $\xi^* H(s)\xi \ge 0$, $\forall \xi \in R^{2n}$; b. *the average of this form over the entire interval is a positive form, i.e.,* $\int_0^S \xi^* H(s)\xi ds > 0, \forall \xi \ne 0$.

It is straightforward to show that the matrix H has eigenvalues at 1 (multiplicity 2) and $\frac{a_1+a_2 \pm \sqrt{(a_1-a_2)^2+4a_0^2}}{2}$, respectively. There exists a positive constant $\hat{\kappa}$, such that for all $\kappa \le \hat{\kappa}$, $H(s)$ is of the positive type (i.e., $a_1 a_2 \ge a_0^2$). In the rest of this section, we first introduce Krien's stability result and then we use it to obtain a sufficient condition for stability of the linearized KV equation.

Lemma 1 (Krein [2]) *All the solutions of the Hamilton system (7) of positive type are stably bounded whenever* $(\omega_1 \ge \omega_2 \ge \cdots \ge \omega_n > 0)$ $\sum_{j=1}^m \frac{\omega_j}{2j-1} < \frac{\pi}{2}$, $m = n/2$, *where* $\pm i\omega_j$ *are the imaginary eigenvalues of the matrix* JH_{av}.

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Corollary 2 *The linearized system (7) is Lyapunov stable if* $\omega_1 + \frac{1}{3}\omega_2 < \frac{\pi}{2}$, *where*

$$\omega_{1,2} = \frac{\sqrt{2\bar{a}_1 + 2\bar{a}_2 \pm 2\sqrt{(\bar{a}_1 - \bar{a}_2)^2 + 4\bar{a}_0^2}}}{2}.$$

Proof: Due to the fact that $\bar{a}_1\bar{a}_2 > \bar{a}_0^2$, the eigenvalues λ_k of JH_{av}, $k = 1, 2, 3, 4$ are pure complex numbers, $\lambda_{1,2} = \pm\frac{i}{2}\omega_1$ and $\lambda_{3,4} = \pm\frac{i}{2}\omega_2$. Therefore, the eigenvalues of the linearized system all reside on the pure complex axis. By using Lemma 1, we have $\omega_1 + \frac{1}{3}\omega_2 < \frac{\pi}{2}$. ■

Remark 1 *This result is easy to check and one does not have to compute the transition matrix to determine system stability, but it is only a sufficient condition for stability and does not give precise information about the boundary of the stability region. Additionally, in order to satisfy the positive type assumption,* $\kappa(s)$ *must be bounded by a value* $\hat{\kappa}$ *that may not be practical.*

STABILITY BY ASYMPTOTIC ANALYSIS

In this section, we focus on the 1D KV equation. We first give a quick review of stability results for 1D linear systems with periodic coefficients, then we apply Simakhina's novel method [3] to study the stability boundaries in terms of the system parameters.

Known Results

The 1D linearized KV equation (3) can be writen as one of the following scalar periodic systems:

$$\text{I}: \quad r'' + p(s)r = 0, \tag{8}$$

$$\text{II}: \quad r'' + [\delta + \varepsilon p(s)]r = 0, \tag{9}$$

which have been extensively studied in the literature. Some of the available results are listed below.

Theorem 3 (Barnes [4]) *The equation (8) is strongly stable, if there exists some integer* n *such that* $p(s) > \frac{n^2\pi^2}{S^2}$ *and* $S\int_0^S p(s)ds < n^2\pi^2 + n\pi^2$.

Remark 2 *Theorem 3 is also a sufficient condition for the stability of the scalar Hill equation (8). Other studies and results are available in [5].*

Theorem 4 (O.Haupt [5]) *For equation (9), the whole* $\varepsilon\delta$*–plane is divided into alternate zones of stability and instability. Let* $K = \frac{1}{2}[r(S) + r''(S)]$, *then the points satisfying* $|K(\delta, \varepsilon)| = 1$ *define curves in the* $\delta\varepsilon$*–plane which separate the plane into regions where the solution of (9) is either stable or unstable.*

Remark 3 *Theorem 4 can provide the critical stability boundaries in the parameter space* (δ, ε), *but it is numerically challenging to compute enough pairs of* (δ, ε) *to generate these stability boundaries.*

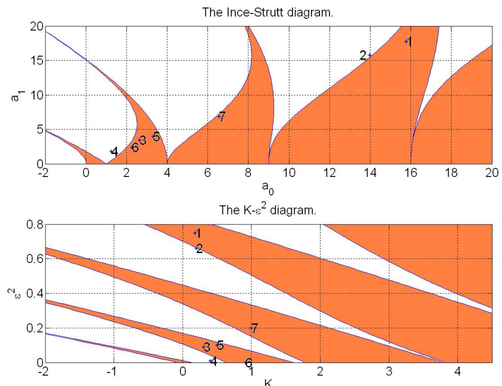

Figure 1: The Ince-Strutt diagram (parameters in the shaded region is the stable) (top); K–ε^2 diagram (bottom).

Stability Regions

In this subsection, we will apply Simakhina's novel method [3] based on asymptotic analysis to consider the stability regions of the 1D linearized KV equation. We consider the linearized form of the 1D KV equation (3),

$$r'' + \psi(s)r = 0,\ \psi = \kappa(s) + 3\varepsilon^2 r_0^{-4}(s) + K r_0^{-2}(s), \tag{10}$$

where the focusing function κ and the matched solutions r_0 are periodic functions of period S. We first consider the Fourier series expression of the periodic coefficient $\psi = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n}{S}s + b_n \sin\frac{2\pi n}{S}s\right)$, where $a_n(\kappa, \varepsilon, K)$ and $b_n(\kappa, \varepsilon, K)$ are Fourier coefficients. We assume $\psi(-s) = \psi(s)$, and truncate the Fourier series to make $\psi \approx \psi_1 = a_0 + a_1 \cos 2s$. Thus, the 1D linearized KV equation becomes

$$r'' + (a_0 + a_1 \cos 2s)\, r = 0. \tag{11}$$

By the Floquet theory (e.g., [5]), the periodic system (11) at least has solutions of period π and 2π. The solution $r(s)$ can be given by Fourier series expansions if the solutions of (10) are all bounded. Let $r = r_1 + r_2$, where $r_1(s) = c_0 + \sum_{n=1}^{\infty} c_n \cos 2ns$ and $r_2(s) = \sum_{n=1}^{\infty} d_n \sin 2ns$. We substitute r and r'' into (11) and we can obtain the following linear equations by considering the linear independency of the different harmonics

$$\begin{cases} a_0 c_0 + \dfrac{a_1}{2} c_1 = 0 \\ a_1 c_0 + (a_0 - 4)c_1 + \dfrac{a_1}{2} c_2 = 0 \\ \dfrac{a_1}{2} c_{n-1} + (a_0 - 4n^2)c_n + \dfrac{a_1}{2} c_{n+1} = 0,\ n \geq 2 \end{cases} \tag{12}$$

$$\begin{cases} (a_0 - 4)d_1 + \dfrac{a_1}{2} d_2 = 0 \\ \dfrac{a_1}{2} d_{n-1} + (a_0 - 4n^2)d_n + \dfrac{a_1}{2} d_{n+1} = 0,\ n \geq 2. \end{cases} \tag{13}$$

For the pure 2π-periodic solution (without π-periodic component), we have $r = \sum_{n=1}^{\infty} c_n \cos(2n-1)s + \sum_{n=1}^{\infty} d_n \sin(2n-1)s$. Substituting r and r'' in (11), we obtain

$$\begin{cases} (a_0 - 1 + \dfrac{a_1}{2})c_1 + \dfrac{a_1}{2} c_2 = 0 \\ \dfrac{a_1}{2} c_{n-1} + (a_0 - (2n-1)^2)c_n + \dfrac{a_1}{2} c_{n+1} = 0,\ n \geq 2 \end{cases} \tag{14}$$

No.	K	ε^2	a_0	a_1	Sim	Ince	Ave
1	.18	.75	15.77	17.72	S	S	F
2	.20	.66	13.99	15.78	S	U	F
3	.30	.09	2.64	3.42	S	S	F
4	.40	.01	1.25	1.82	U	U	F
5	.50	.10	3.32	3.94	S	S	F
6	.91	.00	2.26	2.37	S	S	S
7	1.00	.20	6.54	6.89	S	S	F

Figure 2: Study cases (Sim: simulation; Ince: Ince-Strutt diagram (asymptotic analysis) (Figure 1); Ave: averaging). U denotes "unstable", S denotes "stable", and F denotes "fail to check."

$$\begin{cases} (a_0 - 1 - \frac{a_1}{2})d_1 + \frac{a_1}{2}d_2 = 0 \\ \frac{a_1}{2}d_{n-1} + (a_0 - (2n-1)^2)d_n + \frac{a_1}{2}d_{n+1} = 0,\ n \geq 2. \end{cases} \tag{15}$$

We rewrite the linear equations (12)–(15) into matrix forms: $\mathcal{M}_1\mathbf{c}_1 = 0$, $\mathcal{M}_2\mathbf{c}_2 = 0$, $\mathcal{N}_1\mathbf{d}_1 = 0$, $\mathcal{N}_2\mathbf{d}_2 = 0$, where $\mathcal{M}_1$, $\mathcal{M}_2$ are the coefficient matrices in (12) and (14), and $\mathcal{N}_1$, $\mathcal{N}_2$ are the coefficient matrices in (13) and (15). The unknown vectors are defined as $\mathbf{c}_1 = (c_0, \cdots, c_n, \cdots)^T$, $\mathbf{c}_2 = (c_1, \cdots, c_n, \cdots)^T$, corresponding to (12) and (14), and $\mathbf{d}_1 = \mathbf{d}_2 = (d_1, \cdots, d_n, \cdots)^T$, corresponding to (13) and (15). We can note that the linear equations in (12)–(15) are infinite dimensional. We truncate the linear systems by a finite integer number ($N \geq 3$ in this case) to be able to implement a practical computation of the solution.

In order to ensure the existence of periodic solutions, we obtain the stability boundary equations for (a_0, a_1) by making $\det \mathcal{M}_{1,2} = \det \mathcal{N}_{1,2} = 0$. The stability boundary diagram is shown at the top of Figure 1, which is called the Ince-Strutt diagram in the literature. Systems with parameters (a_0, a_1) in the shaded regions of the Ince-Strutt diagram (Figure 1-top) have bounded periodic solutions. Additionally, the stability boundaries in the Ince-Strutt diagram show very little change for more precise truncations ($N \geq 4$) of the linear systems.

RESULTS

As an example, we let $\kappa = \frac{1}{10} + \cos 2s$ and $r_0 = \frac{1}{\frac{3}{2} + \frac{1}{2}\cos 2s}$. We then compute the Fourier coefficients, $a_0 = \frac{2601}{128}\varepsilon^2 + \frac{1}{10} + \frac{19}{8}K$, $a_1 = \frac{3}{2}K + 1 + \frac{351}{16}\varepsilon^2$ (or $\varepsilon^2 = \frac{608}{5535}a_1 - \frac{2848}{27675} - \frac{128}{1845}a_0$ and $K = \frac{208}{205}a_0 + \frac{2578}{3075} - \frac{578}{615}a_1$). We note that there exists a linear transformation between $(a_0, a_1)^T$ and $(K, \varepsilon^2)^T$. By using this linear transformation, we can obtain the stability regions in terms of the system parameters (K, ε^2) (Figure 1-bottom) based on the Ince-Strutt diagram (Figure 1-top). Several cases were considered to compare the different stability criteria: averaging method, asymptotic analysis, and numerical simulation. These cases are summarized in Figure 2, and also marked in Figure 1. The beam envelope deviation profiles for Case 1 ($K = .18$, $\varepsilon^2 = .75$) and Case 4 ($K = .40$, $\varepsilon^2 = .01$) are shown in Figure 3 and Figure 4 respectively.

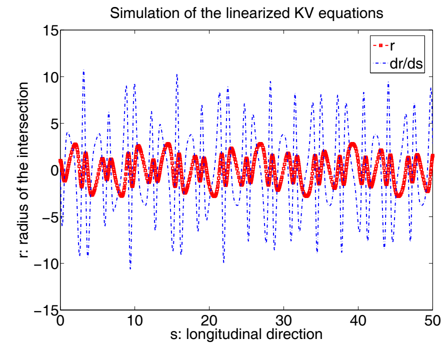

Figure 3: Case 1 ($K = .18$, $\varepsilon^2 = .75$).

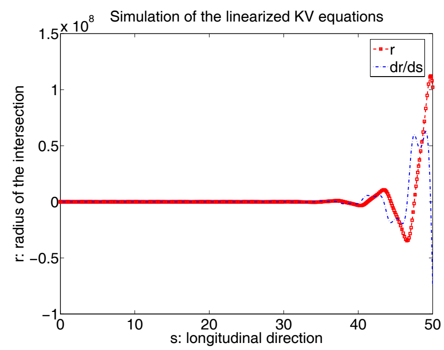

Figure 4: Case 4 ($K = .40$, $\varepsilon^2 = .01$).

CONCLUSIONS

By using Fourier analysis and asymptotic analysis, we can determine the stability of the 1D linearized KV equation. Unfortunately, truncating the Fourier series of the periodic coefficient of the linear KV equation influences the stability conclusion around the boundary of stability (e.g., case 2 in the table above and also in Figure 1). To study stability of the 1D KV equation without truncation of the periodic coefficient, we will consider in the future the integral operator method introduced in [3].

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