NONLINEARITIES IN MAGNETIC CONFINEMENT, IONOSPHERIC PHYSICS AND POPULATION EXPLOSION LEADING TO PROFILE RESILIENCE

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Abstract

Nonlinearities play an important role in many fields. In the field of thermonuclear fusion, it enters into questions such as profile resilience and fluid closure. A nonlinear phenomenon that fusion has in common with planets in astrophysics is the generation of zonal flows. In fusion, these have an outstanding role in setting the level of turbulence and the fluid closure. The role of resonance broadening and the effect of nonlinearities are investigated. The similarities and differences between our systems are discussed with the population explosion and the dynamics of nonlinear systems for drift waves by different states in the profile resilience described with great precision. This reinforces our belief in the drift wave model's broad application, which encompasses current tokamaks, ITER, and the fusion pilot plant.

1. INTRODUCTION

We have recently pointed out that nonlinearities have to be included in the description of several phenomena where they are often left out [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42]. We have here studied the effect of nonlinearities and their impact on Profile resilience [8]. In particular we have looked at the role of resonance broadening since these influences the fluid closure [16, 29, 33, 34], ionospheric physics [22] particle and heat pinches [11, 12, 13, 14] and recent models for the dynamics of fast particles [32, 33]. Although linear theory [15] can be useful to list different types of instabilities, nonlinear theory [1, 2, 3, 4], is always needed to determine the state of saturated turbulence. Initially rather much work was devoted to the separate studies of linear [15] and nonlinear [1, 2, 3, 4] theories. Such work was initially mainly conducted in simple slab geometry. However, most physics relevant for transport is rather small scale and can then be described by the ballooning mode formalism [5]. This description could be used for most phenomena previously described in simple geometry [3, 6].

One of the areas of interest is the fluid closure in magnetized plasmas, where the correct closure was already made in 1988 in the Weiland model [10]. This closure, although initially made intuitively, depends on nonlinear effects through zonal flows and is needed in order to obtain the appropriate pinch effects as shown in Ref. [11]. These are needed both for transport in tokamaks and for understanding particle motion in the ionosphere [6, 9, 23]. Another application for magnetized laboratory plasmas is the elevated dipole [24]. A further application is for the population explosion on earth [31]. The crucial effect to include here is the nonlinear frequency shift [1, 3, 4, 9] which leads to resonance broadening. We note that models for population development [27] do not have the same base as our results for drift waves, while a non-Markovian effect [16], which will show as oscillations of the same type as those for drift waves, is important. The application to tokamak transport, where profiles of density and temperature have been surprisingly insensitive to the exact location of sources, has been named "profile consistency" or "profile resilience" [8]. The basis of our drift wave calculations is the balance between linear growth rate and convective $\mathbf{E}\times\mathbf{B}$ nonlinearity ($\mathbf{v}_{\mathrm{E}}\cdot\nabla\delta T$).

$$\gamma\delta T = \mathbf{v}_{\mathrm{E}}\cdot\nabla\delta T. \tag{1}$$

This leads to the saturation level

$$\frac{e\phi}{T_{\mathrm{e}}} = \frac{\gamma}{\omega_{*\mathrm{e}}}\frac{1}{k_x L_{\mathrm{n}}}, \tag{2}$$

where e is the charge of an electron, ϕ is the electrostatic potential, T_{e} is the temperature of an electron, γ is the mode's growth rate, $\omega_{*\mathrm{e}}$ is the frequency of diamagnetic drift, k_x is the radial propagation factor, and L_{n} is the length of a density gradient scale. The diagonal element for transport can be written as follows:

$$D = \frac{\gamma^3/k_x^2}{\omega_r^2+\gamma^2}. \tag{3}$$

Now it has been found that the fastest growing mode in typical drift wave systems may have negative energy [4]. This means that waves grow when energy is taken from them. This works both linearly (inverse Landau damping) and nonlinearly, with the possibility of explosive instability. An example of inverse Landau damping is the Hammett-Perkins case, as shown by Mattor and Parker [34]. We can also conclude that the nonlinear terms are destabilizing from the figure in Ref. [34]. In such cases, which seem to be typical, we have nonlinear growth [34, 35] which, if fully developed, could lead to an explosive instability [4, 27]. In such cases a nonlinear frequency shift will develop and turn the energy into positive energy, thus stabilizing the system.

We start from a system of three interacting waves j, k, l:

$$\frac{\partial u_{\mathrm{j}}}{\partial t} = \gamma u_{\mathrm{j}} + c_{\mathrm{j}}\, u_{\mathrm{k}}\, u_{\mathrm{l}}\cos\Phi \tag{4a}$$

$$\Phi = \phi_{\mathrm{j}} - \phi_{\mathrm{k}} - \phi_{\mathrm{l}} \tag{4b}$$

$$\frac{\partial \Phi}{\partial t} = \delta\omega + \left(\frac{u_{\mathrm{k}}u_{\mathrm{l}}}{u_{\mathrm{j}}} - \frac{u_{\mathrm{j}}u_{\mathrm{k}}}{u_{\mathrm{l}}} - \frac{u_{\mathrm{j}}u_{\mathrm{l}}}{u_{\mathrm{k}}}\right)\sin\Phi \tag{4c}$$

$$\delta\omega = \sum \alpha_{\mathrm{m}}|u_{\mathrm{m}}|^2 \tag{4d}$$

where $u = e\phi/T_e$. We note that $\delta\omega$ has a higher power in the perturbation. This is the nonlinear frequency shift, which is a change in frequency due to nonlinear interactions between different waves. It will dominate at high levels of perturbation.

The system (4) has a constant of motion.

$$u_{\mathrm{j}}\, u_{\mathrm{k}}\, u_{\mathrm{l}}\sin\Phi + \sum\frac{\alpha_{\mathrm{j}}}{c_{\mathrm{j}}}|u_{\mathrm{j}}|^2 = \Gamma e^{3\gamma t} \tag{5}$$

where Γ is a constant of motion. In order to see qualitatively how the stabilization by a nonlinear frequency shift occurs, we can consider the case of equal amplitudes of perturbations, i.e.,

$u_{\mathrm{j}} = u_{\mathrm{k}} = u_{\mathrm{l}} = u$. Then (5) gives the stationary state:

$$u^3 \sin\Phi + u^4 = \Gamma \tag{6a}$$

$$\kappa = \frac{1}{4}\sum \frac{\alpha_j}{c_j} \tag{6b}$$

We then find the maximum amplitude

$$u = \frac{1}{\kappa} \tag{7}$$

The simplest solution of this system is for $\Gamma = 0$. It then takes the form of a soliton solution in time [4].

$$u(t) = \frac{1}{\sqrt{\kappa^2 + (t_1 - t)^2}} \tag{8a}$$

where

$$t_1 = \frac{1}{u(0)}\sqrt{1 - \kappa^2 u^2(0)} \tag{8b}$$

When $\Gamma \neq 0$, the solution does not meet its initial condition after the maximum, and we get an oscillatory solution. In the general dissipation-free case, we can derive our system from a Hamiltonian.

$$H = \sum_{\mathrm{j}} s_{\mathrm{j}}\,\omega_{\mathrm{j}}\,u_{\mathrm{j}}^2 + 2V\,(u_0^2 u_1^2 u_2^2)^{1/2}\sin\Phi - \sum_{\mathrm{jk}} \gamma_{\mathrm{jk}} u_{\mathrm{j}}^2 u_{\mathrm{k}}^2 \tag{9}$$

For more details, we refer to Ref. [4] pages 135 and 153. However, our main result is that we reach a steady state where we define

$$\cos\Phi = \mu \tag{10}$$

$$u = \frac{\gamma}{c\mu} \tag{11}$$

We are here free to choose c in such a way that (11) fulfills (2) with our present definition of u. This means that our stabilization of a nonlinear instability due to a nonlinear frequency shift maintains our previous saturation level for $\mathbf{E}\times\mathbf{B}$ stabilization of a linear instability. This saturation level has been found to be in good agreement with experiments, as seen from the general agreement of our model with experiment [41, 42].

However, in Refs. [1, 11] we also included the full quasilinear transport, which includes pinch effects. The frequency dependence of (3) is a non-Markovian effect. In order to have pinch effects leading to inward fluxes, we also need nonadiabatic electrons. This was accomplished by including electron trapping, as shown in Ref. [11]. It gave possibility for both particle and thermal pinches. In particular, the particle pinch has been much explored in connection with the elevated dipole [24] and ionospheric phenomena [3, 4]. It was further tested with good agreement for a tokamak experiment (Tore Supra) [27]. Similar agreement was also found with QualiKiz [19, 26] which is a model based on quasilinear theory but fitted to fully nonlinear simulations. We had further experimental support for poloidal spinup in internal transport

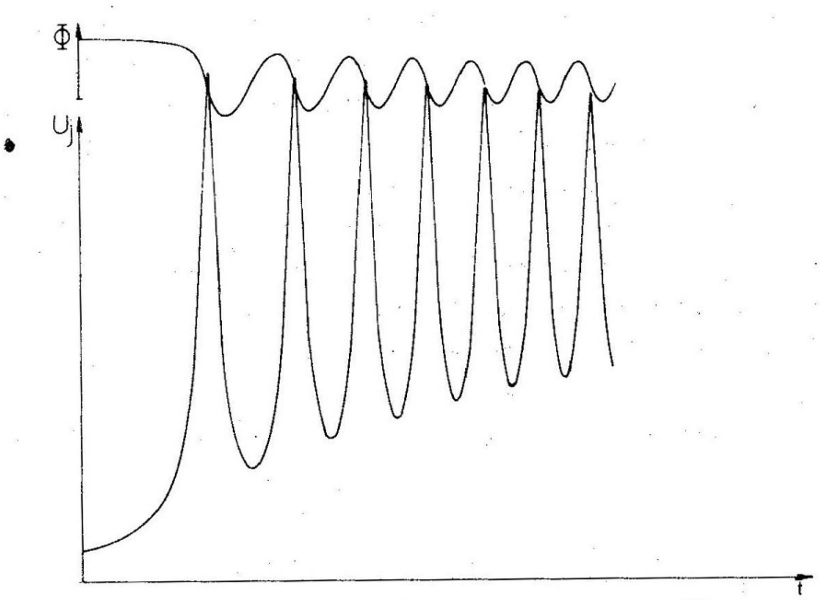

FIG. 1: Stabilization of an explosive instability, from Ref. [4]. The oscillations are due to oscillations in Φ due to an oscillating nonlinear frequency shift. The oscillations in Φ lead to oscillations in cosΦ giving the periodic behaviour of the amplitude.

barriers [20], nonlinear upshift in internal transport barriers, and the L-H transition [23, 43]. We also note that (3) was obtained by Connor and Pogutse for collisional drift waves [17].

A remaining question is the validity of (3) when we have nonlinear growth at low amplitudes, as found in Refs. [34, 35]. This will give rise to the initial phase of explosive instability, as discussed in Ref. [4]. Now we know from Ref. [4] that such an instability will be stabilized by a nonlinear frequency shift, as also seen in Refs. [9, 10]. As seen here, we get oscillations after the initial saturation (compare Refs. [4], [30], and [31]), and this averages out the effect of higher nonlinearities. Here, the nonlinear frequency shift works to change the sign of the wave energy, or dissipation. Thus, we expect (3) to be valid on average. Further, we note that for the population explosion, we only have the initial phase of stabilization by a nonlinear frequency shift. Thus, the growth rate is due to a balance between quadratic and cubic nonlinearity. This becomes very accurate since local and system-dependent quantities usually enter through the linear growth rate.

THE IMPORTANCE OF BOUNDARY CONDITIONS IN K-SPACE

As pointed out above, we have a similarity in the boundary conditions in k-space between Ref. [10] and Ref. [17]. These models both use reactive fluid models. An important aspect of drift wave turbulence is that it cascades both towards shorter and longer wavelengths [6]. The cascade towards shorter wavelengths is a common feature of 3D turbulence, and this is absorbed by viscosity. However, the cascade towards longer wavelengths is critical in a finite system. Here we need a sink for the turbulence, which absorbs the cascade towards long wavelengths [6]. This sink is caused by another nonlinearity, which is the generation of zonal flows [9, 21, 26, 27]. In the absence of this sink, energy is accumulated at the longest wavelengths possible in the system, which leads to excessive transport. Increased transport in the presence of dissipation was already observed in Ref. [9]. Thus, we need strong zonal flows, and these are strongest in reactive systems [10, 17]. Thus, the level of transport depends sensitively on the fluid closure. With a reactive closure we get a fluid description. We note, in particular, the very strong sensitivity of zonal flows to the type of dissipation used in gyrofluid closures [26], where the Dimits nonlinear upshift [21] could not be reproduced when dissipation was included. In a fluid description, a quasilinear description is usually valid (Refs. [17, 31]). However, in a kinetic description, we need the strongly nonlinear resonance broadening. The reason why we need strongly nonlinear effects in a kinetic description but not in a fluid description is the vastly different magnitudes of the velocities. However, in fluid theory, we also need to include zonal flows, as shown in Refs. [20] and [21]. Actually, explicit zonal flows at the correlation length are needed in calculations of Dimits shift, spinup in internal transport barriers, and the L-H transition. When zonal flows are not active at the correlation length, we can just assume an absorbing boundary for long wavelengths, as done in, e.g., Ref. [11].

ORBIT INTEGRATION

The integration along particle orbits has played an important role in the development of our theory. The first example was the derivation of linear and nonlinear gyrokinetic equations [25]. The first nonlinear gyrokinetic equation was actually derived by Frieman and Chen [40] by local orbit averaging. We then derived a nonlinear gyrokinetic equation by orbit integration [25]. This latter derivation is actually shorter, but both derivations lead to the Hasegawa-Mima equation [6] in the appropriate limit. In Ref. [36], we also showed, by orbit integration, that the linear part of the eigenfrequency is typically obtained with high accuracy in a quasilinear treatment, as also found in Ref. [17]. Now, the imaginary part of the frequency can be due either to inhomogeneities in configuration space or wave particle resonances in kinetic theory. The latter reason will vanish in the long time asymptotic part of velocity dispersion, i.e., the deviation of the velocity square from its initial condition. Thus, there will be no more energy transfer between resonant particles and waves.

RESONANCE BROADENING

We have mentioned resonance broadening on several occasions. Resonance broadening [1, 3, 7, 29, 30, 31, 34, 35] occurs due to nonlinear frequency shifts that change the phase velocity of waves so as to take them out of resonance with particles. This process is active when we keep a Maxwellian distribution but still observe that wave particle resonances are not active [34, 35]. This is a strongly nonlinear effect that is not present in quasilinear theory. This effect has recently been added to several studies of fast particle instabilities [32, 33]. We have also recently derived a combined theory for drift wave turbulence and nonlinear friction (resonance broadening), where resonance broadening is one of the main features [39]. As expected, resonance broadening reduces the strength of wave-particle resonances when there are no external sources in velocity space, as usually for drift waves. Resonance broadening is able to make a reactive fluid closure possible [16]. Without resonance broadening, linear Landau damping and magnetic drift resonances remain and are able to completely damp out particle pinches [11, 14]. This is also the reason for the need to make a fit to nonlinear kinetic theory in Ref. [19]. Thus the essence of the fit of QualiKiz to nonlinear kinetic codes is to introduce resonance broadening with an empiric procedure. As mentioned above, this leads to similar results by our model and QualiKiz for the particle pinch in Tore Supra [26, 27]. Thus a strongly nonlinear approach is needed for kinetic theory. On the other hand, orbit integration has shown that the quasilinear approach is sufficient in fluid calculations. This was also found in Ref. [17]. Of course, we define strongly nonlinear as including an explicit nonlinear frequency shift. We also need to include zonal flows in the fluid model [20, 21]. In Ref. [35], we found that there may be a balance between resonance broadening β and an external source in velocity space, S_v for fast particles. Thus, we expect resonance broadening to reduce fast particle instabilities.

$$\left(\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r}\right) W(X,X',t,t') = \frac{\partial}{\partial v}\left[\beta v + D^{v}\frac{\partial}{\partial v}\right] W(X,X',t,t') + S_{\mathrm{v}}. \tag{12}$$

Here W is the transition probability in phase space, β is the nonlinear friction (resonance broadening), D^{v} is the nonlinear diffusivity in velocity space, and S_{v} is an external source (typically heating) in velocity space.

$$\beta = \sum_j \beta_j |\phi_j|^2 \tag{13a}$$

$$D^{v} = \sum_j d_j |\phi_j|^2. \tag{13b}$$

However, at drift wave frequencies, the fast particle source term is typically a factor 100 smaller than resonance broadening and can thus be neglected. We are then back in the Fokker-Planck equation derived in Ref. [16]. Thus, our fluid model is valid to within 1% in the drift wave regime.

For constant coefficients and without source 12 leads to the velocity dispersion.

$$\langle \triangle v^2 \rangle = \frac{D^{v}}{\beta}(1 - e^{-\beta t}). \tag{14}$$

Showing that $\triangle v^2$ saturates for $t > 1/\beta$. This means that after this time, there is no more energy transfer between waves and resonant particles. This fact was tested for much more general cases in Ref. [44].

As pointed out above, resonance broadening is what is needed for kinetic models to give adequate particle pinches [19]. In our fluid model, however, resonance broadening has already turned the kinetic model into a fluid model, and there we do not need further strongly nonlinear effects in order to recover particle pinches [11]. Instead, our quasilinear fluid model allows us to recover the Dimits nonlinear upshift [21, 22, 30], the L-H transition [23], and the poloidal spinup in internal transport barriers [20] when we include zonal flows.

2. DISCUSSION

In conclusion, we have explained the similarity between our system describing the population explosion with extreme accuracy [31] and the dynamics of nonlinear systems for drift waves [9, 10] by different states in the profile resilience [8]. Another important aspect is that orbit integration has shown that our quasilinear fluid approach works extremely well [31, 32, 33, 34, 35]. However, resonance broadening is the main nonlinear effect that turns kinetic theory into fluid theory. Our first derivation of our model was mainly intuitive, assuming that moments without sources in the experiment would be damped out by transport. It was really not until Ref. [16] that we could see how resonance broadening turned a fully nonlinear description into a reactive fluid model. While a quasilinear kinetic model does not have a particle pinch, a fit to a nonlinear kinetic model, introducing resonance broadening [19, 26] recovers the particle pinch. However, our fluid model recovers the particle pinch directly without additional fitting [27]. Furthermore, an important aspect is that we keep $\varepsilon_n = 2L_{\rm n}/R$ arbitrary. This means that we can describe L modes as well as H modes and the L-H transition dynamically [23]. This also includes electromagnetic effects, and the H-mode barrier is typically in the second stability regime of MHD ballooning modes. This strengthens our confidence in the broad applicability of our drift wave model, which includes current tokamaks, ITER, and fusion pilot plants in addition to various ionospheric problems [24], the elevated dipole [24] and the population explosion [31].

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