Mixing Enhancement by Boundary Feedback Control in 2D MHD Channel Flow: A Numerical Study

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A direct numerical simulation (DNS) code has been developed, based on a hybrid Fourier pseudospectral-finite difference discretization scheme and the fractional step technique, to simulate a two-dimensional (2D) magnetohydrodynamic (MHD) channel flow, also known as Hartmann flow. This flow is characterized by an electrically conducting, incompressible fluid moving between parallel plates in the presence of an externally imposed transverse magnetic field. The system is described by the MHD equations, a combination of the Navier-Stokes equation and the Magnetic Induction equation, which is derived from the Maxwell equations. The laminarization effect of the imposed magnetic field is studied numerically. In addition, a nonlinear, Lyapunov-based, boundary feedback control law designed for mixing enhancement is numerically tested. Pressure sensors, magnetic field sensors, and micro-jets embedded into the walls of the flow domain are considered in this work to implement a feedback control law that maximizes a measure related to mixing (which incorporates stretching and folding of material elements), and minimizes the control and sensing efforts.

I. Introduction

While control of flows has been an active area for several years now, up until now active feedback flow control developments have had little impact on electrically conducting fluids moving in electromagnetic fields. Active feedback control in electrically conducting flows, implemented through micro-electro-mechanical or micro-electro-magnetic actuators and sensors, can be used to achieve optimally the desired level of stability (when suppression of turbulence is desired) or instability (when enhancement of mixing is desired). As a result, a small amount of active control applied to magnetohydrodynamic (MHD) flows, magnetogasdynamic (MGD) flows, and plasma flows can dramatically change their equilibrium profiles and stability (turbulence fluctuations) properties. These changes influence heat transfer, hydrodynamic drag, pressure drop, and the required pumping power for driving the fluid.

Prior work in the area of active control of electrically-conducting-fluid flows focuses mainly on electro-magneto-hydro-dynamic (EMHD) flow control for hydrodynamic drag reduction, through turbulence control, in weak electrically conducting fluids such as saltwater. Traditionally two types of actuator designs have been used: one type generates a Lorentz field parallel to the wall in the streamwise direction, while the other type generates a Lorentz field normal to the wall in the spanwise direction. EMHD flow control has been dominated by strategies that either permanently activate the actuators or pulse them at arbitrary frequencies. However, it has been show that feedback control schemes, making use of "ideal" sensors, can improve the efficiency, by reducing control power, for both streamwise[1] and spanwise[2,3] approaches. From a model-based-control point of view, it is worth to mention that feedback controllers for drag reduction have been previously designed using distributed control techniques based on linearization and model reduction.[4,5]

In this paper, we consider a Hartmann flow, an electrically conducting fluid moving between parallel plates through an imposed transverse magnetic field. The fluid is considered incompressible and Newtonian (constant viscosity). When an electrically conducting fluid moves in the presence of a transverse magnetic field, it produces and electrical field due to charge separation and subsequently an electric current. The interaction between this

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created electric current and the imposed magnetic field originates a body force, called the Lorentz force, which acts on the fluid itself. Since this force acts in the opposite direction of the fluid motion, it is necessary to increase the pressure drop to maintain the mean velocity of the flow and a considerable increase of power is required to pump the fluid. In addition, this force tends to suppress turbulence and laminarize the flow, reducing the heat transfer rate as consequence. The interaction between a magnetic field and an electrically-conducting-fluid flow can have a positive or a negative impact on mixing and heat transfer depending on the application.

The high temperature reached at the surface of a vehicle flying at hypersonic speed causes the ionization of the surrounding air molecules and the consequent formation of a plasma. It is natural then to try to exploit the plasma capability of interacting with an electromagnetic field. By imposing a suitable magnetic field, it is possible to modify the aerodynamic forces and heat transfer rates in a convenient way. Since the Lorentz force tends to oppose fluid motion across magnetic field lines, a transverse magnetic field applied to the plasma layer would tend to increase the drag, braking the vehicle in atmospheric entry, and to reduce heat transfer and skin friction by slowing and laminarizing the flow near the surface of the body. The use of this MHD principle would have been an alternative to the use of heat shield, the conventional approach to thermal protection for the last 40 years. But these ideas could not be put in practice because of the large, heavy magnets required to provide a magnetic field strong enough to affect the thermally ionized reentry flow characterized by a relatively low electrical conductivity. This subject has been revived by the appearance of superconducting magnets. It seems that the light-weight magnets, together with the utilization of artificial ionization, would give some new consideration to the usage of electromagnetic control techniques. Present work studies numerically the influence of the imposed magnetic field in different flows.[6–10] However, the feedback of some information of the system, such as variations in the current density, induced magnetic field or pressure, has not been considered to modulate the intensity of the imposed magnetic field. In this case, active feedback control can be used to modulate intelligently small magnets (weak magnetic field) instead of using large, heavy, unmodulated large magnets (strong magnetic field) to achieve the same degree of laminarization and heat transfer mitigation, minimizing the control power at the same time.

In cooling systems, since the Lorentz force acts in the opposite direction of the fluid motion, it is necessary to increase the pressure drop to maintain the mean velocity of the flow. In addition, an increase of power is required to pump the liquid through the ducts forming the cooling system. The heat transfer decrease due to the laminarization of the flow prevents present cooling systems (liquid metals (bismuth, gallium, lithium, sodium-potassium, and some of their alloys), electrically conducting liquid salts (Flibe),[11] and nanofluids[12] (tiny spherical copper particles of a few nanometers are added to conventional fuids)) from producing the heat transfer improvements which might be expected based on the much higher thermal conductivity of the coolant. In this case, active control can be used to enhance turbulence, mixing, and therefore heat transfer. As a result, a small amount of active control can greatly influence the heat transfer characteristics of the cooling system or the external power needed to sustain its operation. The possible usage of liquid metals or electrically conducting liquid salts (Flibe) as self-cooled blankets in magnetic confinement fusion reactors has been in consideration for the past 30 years. The main function of the coolant is the absorption of energy from the neutron flux and the transfer of heat to an external energy conversion system. In addition, if a breeder liquid metal such as liquid-lithium is considered, the blanket can also carry out the breeding of tritium, which is part of the fuel used by the reactor. The liquid metal flow is affected by the strong magnetic field (5-10 Tesla) used to confine the plasma inside the reactor. The interaction between the flow and the strong imposed magnetic field generates very intense magnetohydrodynamic (MHD) effects. Among the most important effects, we find the increase of pressure drop and the decrease of heat transfer rate. A good review of the present state of research in the field can be found in previous work.[11,13] It is possible to note that although extensive experimental and numerical work is going on, the use of feedback to increase the heat transfer rate without increasing the pressure drop has not yet been explored. The usage of liquid-metal-based cooling systems is also currently under consideration in the computer industry. The need for cooling solutions for semiconductor devices has never been greater than today. With higher power dissipation due to higher speed processors, the demand for advanced cooling solutions will continue growing. The thermal and physical properties of liquid metals put them in advantage over other single phase liquid solutions considered for computing systems. The high boiling point and thermal conductivity make the liquid metal ideal for heat removal and dissipation. Although supercomputers under development nowadays will produce huge amount of heat to be removed, this is not the only market for liquid-metal cooling systems. The development of new generation of portable and desktop computer is also restricted by heat removal limitations. A new technology that enables increase of heat transfer and decrease of pumping power will have a tremendous economical impact by reducing the power consumption of the cooling system. In addition, it will allow computing systems to keep growing in computational power and speed.

In this work, we focus on the latter application. We use feedback boundary control to improve mixing by enhancing the instability of the Hartmann flow profile. From the point of view of sensors and actuators we follow the ideas introduced in previous work.[14–17] Micro-jets, pressure sensors, and magnetic field sensors embedded into the walls of the flow domain are used to implement the feedback control law. The mathematical foundation of the proposed control law has been previously introduced,[18] but it is also included in this paper for completeness. This paper focuses on the analysis of the effectiveness of the proposed optimal controller based on numerical simulations. With this purpose, a direct numerical simulation (DNS) code has been developed, based on a hybrid Fourier pseudospectral-finite difference discretization scheme and the fractional step technique, to simulate a full magnetohydrodynamic (MHD) channel flow, also known as Hartmann flow.

The paper is organized as follow. Section II introduces the governing equations of our system. The fully developed solution is presented in Section III and the perturbation equations are introduced in Section IV. The Lyapunov analysis for the designed boundary control law is presented in Section V. In Section VI the numerical method used to simulate the MHD channel flow is briefly described. Results from simulation studies are presented in Section VII. Section VIII closes the paper stating the conclusions and the identified future work.

II. Governing Equations

Let us consider the flow of an incompressible, conducting fluid between parallel plates where a magnetic field $\mathbf{B_o} = B_o\hat{\mathbf{y}}$ perpendicular to the channel axis is externally applied. In addition, let us assume the presence of a uniform pressure gradient in the $-\hat{\mathbf{x}}$ direction. Figure 1 illustrates the configuration. This flow was first investigated experimentally and theoretically by Hartmann.[19]

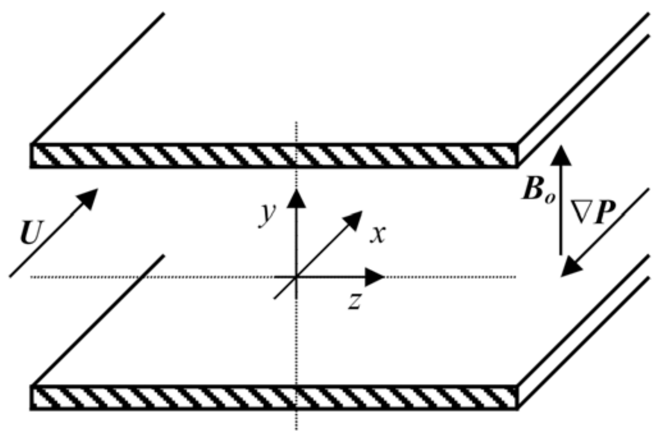


Figure 1. Flow between parallel plates in the presence of a transverse magnetic field (Hartmann flow).

The governing equations for the stated problem are the transport equation of linear momentum

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla P + \rho\nu\nabla^2\mathbf{v} + \mathbf{f} + \mathbf{j}\times\mathbf{B}, \tag{1}$$

and the transport equation of magnetic induction

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} = \frac{1}{\mu\sigma}\nabla^2\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{v}. \tag{2}$$

The flow velocity is denoted by $\mathbf{v}$, the magnetic field by $\mathbf{B}$ and the current density by $\mathbf{j}$, while P denotes the pressure, ρ the fluid mass density, ν the kinematic viscosity, μ the magnetic permeability and σ the electrical conductivity. The volumetric forces of non-electromagnetic origin is represented by $\mathbf{f}$ and the $\mathbf{j}\times\mathbf{B}$ term represents the Lorentz forces. The Lorentz forces couple the mechanical and electrodynamic states of the system and act in planes perpendicular to both current density and magnetic field vectors. Coulomb forces $q\mathbf{E}$, where q is the electrical charge and $\mathbf{E}$ the electrical field, are negligible in comparison to the Lorentz forces.

The magnetic induction equation is derived in Appendix A from Ohm's law

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \tag{3}$$

Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{4}$$

Ampere's law

$$\mu \mathbf{j} = \nabla \times \mathbf{B}, \tag{5}$$

and the fact that $\mathbf{B}$ and $\mathbf{v}$ are solenoidal

$$\nabla \cdot \mathbf{B} = 0, \tag{6}$$
$$\nabla \cdot \mathbf{v} = 0. \tag{7}$$

In this work we consider the 2D Hartmann flow. Figure 2 shows the geometrical arrangement, where $-L \leq y \leq L$, $-\infty < x < \infty$. The imposed magnetic field $\mathbf{B_o}$ is perpendicular to both planes. In this case we can write $\mathbf{v} = \mathbf{v}(x, y, t) = U(x, y, t)\hat{\mathbf{x}} + V(x, y, t)\hat{\mathbf{y}}$, $\mathbf{B} = \mathbf{B}(x, y, t) = B^u(x, y, t)\hat{\mathbf{x}} + B^v(x, y, t)\hat{\mathbf{y}}$ and $P = P(x, y, t)$.

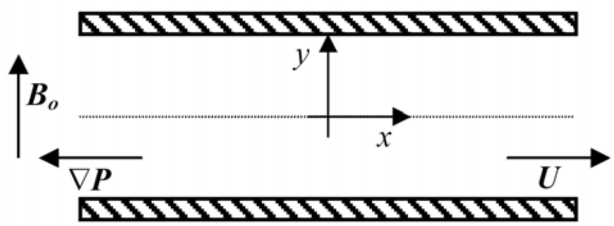


Figure 2. 2D Hartmann flow.

III. Equilibrium Solution

For channels with constant cross section, as the one depicted in Figure 2, a fully developed equilibrium flow is established. In this case, the flow velocity $\bar{\mathbf{v}} = \bar{U}(y)\hat{\mathbf{x}}$ has only one component, which depends on the coordinate y (the upper bar denotes equilibrium variables). The magnetic field is decomposed into two contributions, one due to the external imposed magnetic field and the other caused by the magnetic field induced by the flow $\bar{\mathbf{B}} = \mathbf{B_o} + \bar{\mathbf{b}} = B_o\hat{\mathbf{y}} + \bar{\mathbf{b}}$. Substituting this expression for the equilibrium magnetic field $\bar{\mathbf{B}}$ into equation (2), and forcing the temporal derivative to zero, shows that the only component of the induced magnetic field is $\bar{\mathbf{b}} = \bar{b}(y)\hat{\mathbf{x}}$. The induction equation reduces then to

$$0 = \mu\sigma B_o \frac{d\bar{U}}{dy} + \frac{d^2\bar{b}}{dy^2}. \tag{8}$$

Using Ampere's law (5) it is possible to write the current density $\mathbf{j}$, and consequently the Lorentz force $\mathbf{j} \times \mathbf{B}$, in terms of $\bar{b}$. Then the momentum equation can be written as

$$0 = -\frac{d\bar{P}}{dx} + \frac{B_o}{\mu}\frac{d\bar{b}}{dy} + \rho\nu\frac{d^2\bar{U}}{dy^2}. \tag{9}$$

We consider viscous fluids with no slip at the fluid-wall interface Γ. Therefore the hydrodynamic boundary condition is

$$\bar{\mathbf{v}} = 0 \quad \text{at} \;\; \Gamma, \tag{10}$$

which means that all the velocity components vanish at the wall. For walls with finite electrical conductivity σ_w, magnetic permeability μ_w and normal $\mathbf{n}$, the condition that the tangential component of the electrical field is continuos across the wall interface can be expressed in terms of $\bar{b}$ as[20]

$$\frac{\partial \bar{b}}{\partial n} - \frac{1}{c}\bar{b} = 0 \quad \text{at}\,\Gamma, \tag{11}$$

with the wall conductance ratio defined as $c = \frac{\mu_w \sigma_w t_w}{\mu \sigma L}$ where the wall thickness t_w is often small compared to the dimension of the cross section L. Two limiting cases can be considered

$$\begin{array}{lll} \bar{b} = 0 & \text{at}\,\Gamma\,\text{as}\,c \to 0 & \text{(perfectly insulating walls)} \\ \frac{\partial \bar{b}}{\partial n} = 0 & \text{at}\,\Gamma\,\text{as}\,c \to \infty & \text{(perfectly conducting walls).} \end{array} \tag{12}$$

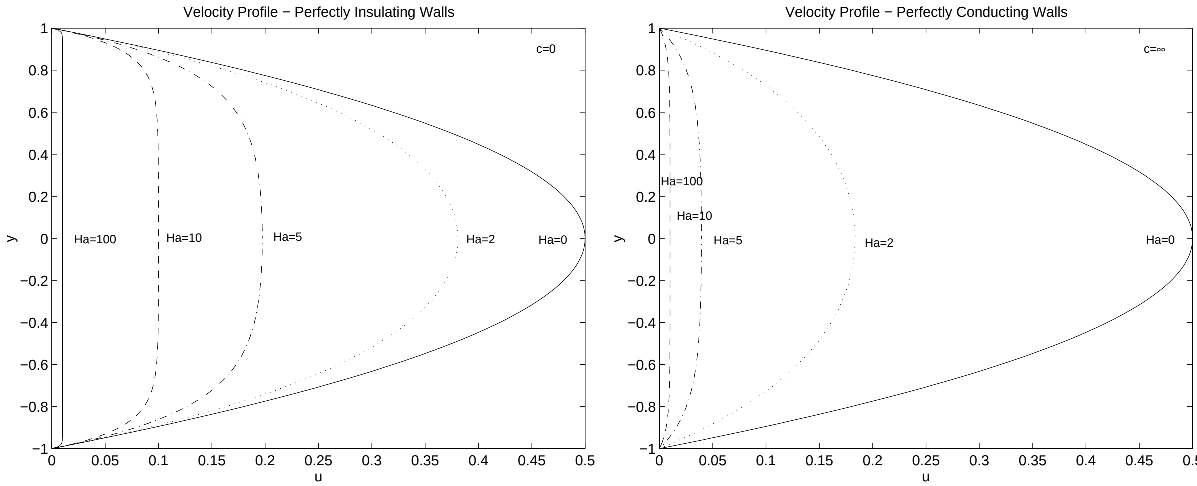


Figure 3. Velocity profiles for Hartmann flow at Hartmann numbers $Ha = 0, 2, 5, 10, 100$ for perfectly insulating walls ($c = 0$) and for perfectly conducting walls ($c = \infty$).

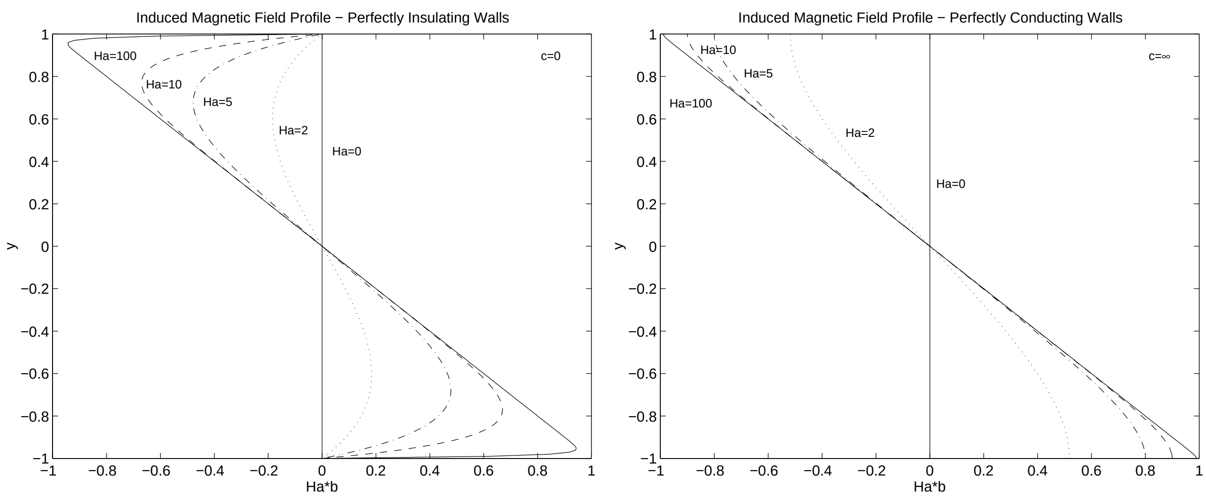


Figure 4. Induced magnetic field profiles for Hartmann flow at Hartmann numbers $Ha = 0, 2, 5, 10, 100$ for perfectly insulating walls ($c = 0$) and for perfectly conducting walls ($c = \infty$).

Defining the dimensionless variables

$$y^* = \frac{y}{y_o} \tag{13}$$

$$\bar{U}^* = \frac{\bar{U}}{U_o} \tag{14}$$

$$\bar{b}^* = \frac{\bar{b}}{b_o} \tag{15}$$

where $y_o = L$, $U_o = \frac{L^2}{\rho\nu}(-\frac{\partial \bar{P}}{\partial x})$, and $b_o = \mu L^2 \sqrt{\frac{\sigma}{\rho\nu}}(-\frac{\partial \bar{P}}{\partial x})$, we can rewrite equations (8) and (9) as

$$Ha\frac{d\bar{U}^*}{dy^*} + \frac{d^2\bar{b}^*}{dy^{*2}} = 0 \tag{16}$$

$$Ha\frac{d\bar{b}^*}{dy^*} + \frac{d^2\bar{U}^*}{dy^{*2}} = -1 \tag{17}$$

with boundary conditions (10) and (11) now expressed as

$$\begin{array}{rclr} \bar{U}^* & = & 0 & \text{at}\, y^* = \pm 1 \\ \mp \frac{d\bar{b}^*}{dy^*} - \frac{\bar{b}^*}{c} & = & 0 & \text{at}\, y^* = \pm 1 \end{array} \tag{18}$$

where $Ha = B_o L \sqrt{\frac{\sigma}{\rho\nu}}$ is the Hartmann number. The solution for system (16) – (17) with boundary conditions (18) is given by

$$\bar{U}^*(y^*) = \frac{1}{Ha}\frac{c+1}{cHa + \tanh(Ha)}\left[1 - \frac{\cosh(Ha\, y^*)}{\cosh(Ha)}\right] \tag{19}$$

$$\bar{b}^*(y^*) = -\frac{y^*}{Ha} + \frac{1}{Ha}\frac{c+1}{cHa + \tanh(Ha)}\frac{\sinh(Ha\, y^*)}{\cosh(Ha)}. \tag{20}$$

Figures 3 and 4 show respectively the velocity and induced magnetic field profiles for different values of the Hartmann number Ha and for perfectly insulating walls, $c = 0$, and perfectly conducting walls, $c = \infty$.

IV. Perturbation Equations

The dimensionless versions of equations (1) and (2) are given in Appendix B. Without loss of generality, we consider in this work that there are no non-electromagnetic volumetric forces present in the system ($\mathbf{f} = 0$). Dropping the star notation we can write the dimensionless momentum and induction equations as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P + \frac{1}{R}\nabla^2 \mathbf{v} + \frac{N}{R_m}\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right] \tag{21}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} = \frac{1}{R_m}\nabla^2\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{v}. \tag{22}$$

Defining the deviation variables as

$$\begin{array}{rcl} u & = & U - \bar{U} \\ v & = & V - \bar{V} = V \\ b^u & = & B^u - \bar{B}^u = B^u - \bar{b} \\ b^v & = & B^v - \bar{B}^v = B^v - B_o \\ p & = & P - \bar{P} \end{array}$$

we can write the dimensionless perturbation equations as

$$\frac{\partial u}{\partial t} + (\bar{U}+u)\frac{\partial u}{\partial x} + v\frac{\partial(\bar{U}+u)}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{N}{R_m}(B_o + b^v)\left(\frac{\partial b^v}{\partial x} - \frac{\partial b^u}{\partial y}\right) + \frac{N}{R_m} b^v \frac{\partial \bar{b}}{\partial y} \tag{23}$$

$$\frac{\partial v}{\partial t} + (\bar{U}+u)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{N}{R_m}(\bar{b} + b^u)\left(\frac{\partial b^v}{\partial x} - \frac{\partial b^u}{\partial y}\right) - \frac{N}{R_m} b^u \frac{\partial \bar{b}}{\partial y} \tag{24}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{25}$$

$$
\begin{aligned}
\frac{\partial b^u}{\partial t} + (\bar{U}+u)\frac{\partial b^u}{\partial x} + v\frac{\partial(\bar{b}+b^u)}{\partial y} &= \frac{1}{R_m}\left(\frac{\partial^2 b^u}{\partial x^2} + \frac{\partial^2 b^u}{\partial y^2}\right) \\
&\quad +(\bar{b}+b^u)\frac{\partial u}{\partial x} + (B_o + b^v)\frac{\partial u}{\partial y} + b^v\frac{\partial U}{\partial y} \qquad (26)\\
\frac{\partial b^v}{\partial t} + (\bar{U}+u)\frac{\partial b^v}{\partial x} + v\frac{\partial b^v}{\partial y} &= \frac{1}{R_m}\left(\frac{\partial^2 b^v}{\partial x^2} + \frac{\partial^2 b^v}{\partial y^2}\right) \\
&\quad +(\bar{b}+b^u)\frac{\partial v}{\partial x} + (B_o + b^v)\frac{\partial v}{\partial y} \qquad (27)\\
\frac{\partial b^u}{\partial x} + \frac{\partial b^v}{\partial y} &= 0. \qquad (28)
\end{aligned}
$$

with initial conditions $u(x,y,0) = u_o(x,y)$, $v(x,y,0) = v_o(x,y)$, $b^u(x,y,0) = b^u_o(x,y)$, $b^v(x,y,0) = b^v_o(x,y)$ for $-\infty < x < \infty$, $-1 < y < 1$ and $t > 0$.

V. Energy Analysis

Choosing the energy function

$$
E(\mathbf{v},\mathbf{B}) = \frac{1}{2}\int_{-1}^{1}\int_{0}^{d} k_1(u^2+v^2) + k_2({b^u}^2 + {b^v}^2)dxdy, \qquad (29)
$$

we can compute

$$
\begin{aligned}
\dot{E}(\mathbf{v},\mathbf{B}) &= \int_{-1}^{1}\int_{0}^{d}(k_1 u u_t + k_1 v v_t + k_2 b^u b^u_t + k_2 b^v b^v_t)dxdy \\
&= k_1\int_{-1}^{1}\int_{0}^{d} -u\left[\bar{U}u_x + uu_x + v\bar{U}' + vu_y - \frac{1}{R}\left(u_{xx}+u_{yy}\right) + p_x\right]dxdy \qquad (30)\\
&\quad +k_1\int_{-1}^{1}\int_{0}^{d} -u\left[\frac{N}{R_m}B_o\left(b^v_x - b^u_y\right) + \frac{N}{R_m}b^v\left(b^v_x - b^u_y\right) - \frac{N}{R_m}b^v\bar{b}'\right]dxdy \qquad (31)\\
&\quad +k_1\int_{-1}^{1}\int_{0}^{d} -v\left[\bar{U}v_x + uv_x + vv_y - \frac{1}{R}\left(v_{xx}+v_{yy}\right) + p_y\right]dxdy \qquad (32)\\
&\quad +k_1\int_{-1}^{1}\int_{0}^{d} -v\left[-\frac{N}{R_m}\bar{b}\left(b^v_x - b^u_y\right) - \frac{N}{R_m}b^u\left(b^v_x - b^u_y\right) + \frac{N}{R_m}b^u\bar{b}'\right]dxdy \qquad (33)\\
&\quad +k_2\int_{-1}^{1}\int_{0}^{d} -b^u\left[\bar{U}b^u_x + ub^u_x + v\bar{b}' + vb^u_y - \frac{1}{R_m}\left(b^u_{xx}+b^u_{yy}\right)\right]dxdy \qquad (34)\\
&\quad +k_2\int_{-1}^{1}\int_{0}^{d} -b^u\left[-\bar{b}u_x - b^u u_x - B_o u_y - b^v u_y - b^v\bar{U}'\right]dxdy \qquad (35)\\
&\quad +k_2\int_{-1}^{1}\int_{0}^{d} -b^v\left[\bar{U}b^v_x + ub^v_x + vb^v_y - \frac{1}{R_m}\left(b^v_{xx}+b^v_{yy}\right)\right]dxdy \qquad (36)\\
&\quad +k_2\int_{-1}^{1}\int_{0}^{d} -b^v\left[-\bar{b}v_x - b^u v_x - B_o v_y - b^v v_y\right]dxdy., \qquad (37)
\end{aligned}
$$

where $\bar{U}'$ and $\bar{b}'$ denote $\bar{U}_y$ and $\bar{b}_y$ respectively. We assume periodic boundary conditions in the streamwise direction, i.e., $\mathbf{v}(x=0) = \mathbf{v}(x=d)$, $\mathbf{B}(x=0) = \mathbf{B}(x=d)$ and $P(x=0) = P(x=d)$. In addition we apply control in the wall normal direction

$$
\begin{aligned}
u(x,-1,t) &= u(x,1,t) = 0 \qquad (38)\\
v(x,-1,t) &= v(x,1,t) = v_{wall}(x,t), \qquad (39)
\end{aligned}
$$

and measure the wall normal component of the induced magnetic field

$$
\begin{aligned}
b^u(x,-1,t) &= b^u(x,1,t) = 0 \qquad (40)\\
b^v(x,-1,t) &= b^v_{bot\_wall}(x,t), \quad b^v(x,1,t) = b^v_{top\_wall}(x,t). \qquad (41)
\end{aligned}
$$

Lemma 1 *Taking into account boundary conditions (38)–(41) the time derivative of $E(\mathbf{v},\mathbf{B})$ along the trajectories can be written as*

$$\dot{E}(\mathbf{v},\mathbf{B}) = -\frac{1}{R}m(\mathbf{v},\mathbf{B}) - \int_0^d v_{wall}\left[k_1\Delta p + k_2\frac{\Delta\left(b^{v^2}\right)}{2}\right]dx + g(\mathbf{v},\mathbf{B}), \tag{42}$$

where

$$\begin{aligned}
m(\mathbf{v},\mathbf{B}) &= k_1\int_{-1}^{1}\int_0^d (u_x^2+u_y^2+v_x^2+v_y^2)dxdy \\
&+ k_2\frac{R}{R_m}\int_{-1}^{1}\int_0^d \left((b_x^u)^2+(b_y^u)^2+(b_x^v)^2+(b_y^v)^2\right)dxdy, && (43)\\
g(\mathbf{v},\mathbf{B}) &= -k_1\int_{-1}^{1}\int_0^d \bar{U}'uvdxdy && (44)\\
&- k_2\int_{-1}^{1}\int_0^d \bar{b}'b^u v dxdy && (45)\\
&+ k_2\int_{-1}^{1}\int_0^d \bar{U}'b^u b^v dxdy && (46)\\
&+ k_1\int_{-1}^{1}\int_0^d \frac{N}{R_m}\bar{b}'\left(ub^v - vb^u\right)dxdy && (47)\\
&+ k_1\int_{-1}^{1}\int_0^d \frac{N}{R_m}\bar{b}\left(b_x^v - b_y^u\right)vdxdy && (48)\\
&- k_1\int_{-1}^{1}\int_0^d \frac{N}{R_m}B_o\left(b_x^v - b_y^u\right)udxdy && (49)\\
&+ k_2\int_{-1}^{1}\int_0^d \bar{b}\left(b^u u_x + b^v v_x\right)dxdy && (50)\\
&+ k_2\int_{-1}^{1}\int_0^d B_o\left(b^u u_y + b^v v_y\right)dxdy && (51)\\
&+ k_1\int_{-1}^{1}\int_0^d \frac{N}{R_m}b^u\left(b_x^v - b_y^u\right)vdxdy && (52)\\
&- k_1\int_{-1}^{1}\int_0^d \frac{N}{R_m}b^v\left(b_x^v - b_y^u\right)udxdy && (53)\\
&+ k_2\int_{-1}^{1}\int_0^d b^u b^u u_x dxdy && (54)\\
&+ k_2\int_{-1}^{1}\int_0^d b^u b^v v_x dxdy && (55)\\
&+ k_2\int_{-1}^{1}\int_0^d b^v b^u u_y dxdy && (56)\\
&+ k_2\int_{-1}^{1}\int_0^d b^v b^v v_y dxdy, && (57)
\end{aligned}$$

and

$$\begin{aligned}
\Delta p &= p(x,1,t) - p(x,-1,t) = P(x,1,t) - P(x,-1,t) && (58)\\
\Delta\left(b^{v^2}\right) &= (b^v(x,1,t))^2 - (b^v(x,-1,t))^2 = b^{v^2}(x,1,t) - b^{v^2}(x,-1,t). && (59)
\end{aligned}$$

Proof. See Appendix C. □

The stretching of material elements accompanied by folding are keys to effective mixing. The measure (43) seems to be strongly connected to mixing since there is a direct relation between stretching of material elements and the spatial gradients of the flow field. Folding is present implicitly in (43) due to the boundedness of the flow domain and the fact that $\mathbf{v}$ satisfies the Navier-Stokes equation.

Lemma 2 *The function $g(\mathbf{v},\mathbf{B})$ satisfies*

$$|g(\mathbf{v},\mathbf{B})| \leq g_1 m(\mathbf{v},\mathbf{B}) + g_2 m^2(\mathbf{v},\mathbf{B}) + g_3 \int_0^d v_{wall}^2 dx + g_4 \int_0^d (b^v_{top\_wall})^2 dx + g_5 \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2$$

where g_1, g_2, g_3, g_4 and g_5 are constants conveniently defined.

Proof. See Appendix D. □

The design goal is a feedback control law, in terms of suction and blowing of fluid normally to the channel wall, that is optimal with respect to some meaningful cost functional related to $m(\mathbf{v},\mathbf{B})$.

Theorem 1 *The cost functional*

$$J(v_{wall}) = \lim_{t\to\infty} \left[2\beta E(\mathbf{v}(t),\mathbf{B}(t)) + \int_0^t h(\mathbf{v}(\tau),\mathbf{B}(\tau)) d\tau \right] \quad (60)$$

where

$$\begin{aligned} h(\mathbf{v},\mathbf{B}) &= \frac{2\beta}{R} m(\mathbf{v},\mathbf{B}) - 2\beta \left[g(\mathbf{v},\mathbf{B}) + g_4 \int_0^d (b^v_{top\_wall})^2 dx + g_5 \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2 \right] \\ &\quad -\beta \int_0^d v_{wall}^2 dx - \beta \int_0^d \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right]^2 dx \end{aligned} \quad (61)$$

is maximized by the control

$$v_{wall} = -\left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right]. \quad (62)$$

Moreover, solutions of system (23)–(28) satisfy

$$h(\mathbf{v},\mathbf{B}) \leq l_1 m(\mathbf{v},\mathbf{B}) + l_2 m^2(\mathbf{v},\mathbf{B}) - l_3 \int_0^d v_{wall}^2 dx - \beta \int_0^d \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right]^2 dx \quad (63)$$

for arbitrary values of control v_{wall} and with

$$l_1 = 2\beta \left(\frac{1}{R} + g_1 \right), \quad l_2 = 2\beta g_2, \quad l_3 = \beta - g_3. \quad (64)$$

Proof. By Lemma 1, we can write equation (61)

$$\begin{aligned} h(\mathbf{v},\mathbf{B}) &= \left(-2\beta \dot{E}(\mathbf{v},\mathbf{B}) - 2\beta \int_0^d v_{wall} \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right] dx + 2\beta g(\mathbf{v},\mathbf{B}) \right) - 2\beta g(\mathbf{v},\mathbf{B}) \\ &\quad -2\beta g_4 \int_0^d (b^v_{top\_wall})^2 dx - 2\beta g_5 \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2 \\ &\quad -\beta \int_0^d v_{wall}^2 dx - \beta \int_0^d \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right]^2 dx \end{aligned}$$

$$
\begin{aligned}
&= -2\beta \dot{E}(\mathbf{v},\mathbf{B}) - 2\beta g_4 \int_0^d (b^v_{top\_wall})^2 dx - 2\beta g_5 \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2 \\
&\quad -\beta \int_0^d v_{wall}^2 dx - 2\beta \int_0^d v_{wall} \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right] dx - \beta \int_0^d \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right]^2 dx \\
&= -2\beta \dot{E}(\mathbf{v},\mathbf{B}) - 2\beta g_4 \int_0^d (b^v_{top\_wall})^2 dx - 2\beta g_5 \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2 \\
&\quad -\beta \int_0^d \left(v_{wall} + \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right] \right)^2 dx
\end{aligned} \tag{65}
$$

and the cost functional can be written as

$$
\begin{aligned}
J(v_{wall}) &= \lim_{t\to\infty} \Bigg[2\beta E(\mathbf{v}(t),\mathbf{B}(t)) - 2\beta \int_0^t \dot{E}(\mathbf{v}(\tau),\mathbf{B}(\tau)) d\tau \\
&\quad -2\beta g_4 \int_0^t \int_0^d (b^v_{top\_wall})^2 dx d\tau - 2\beta g_5 \int_0^t \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2 d\tau \\
&\quad -\beta \int_0^t \int_0^d \left(v_{wall} + \left[\Delta p + \frac{\Delta \left(b^{v^2} \right)}{2} \right] \right)^2 dx d\tau \Bigg] \\
&= 2\beta E(\mathbf{v}(0),\mathbf{B}(0)) - 2\beta \lim_{t\to\infty} \left[g_4 \int_0^t \int_0^d (b^v_{top\_wall})^2 dx d\tau + g_5 \int_0^t \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2 d\tau \right] \\
&\quad -\beta \lim_{t\to\infty} \int_0^t \int_0^d \left(v_{wall} + \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right] \right)^2 dx d\tau.
\end{aligned} \tag{66}
$$

The cost functional (60) is maximized when the last integral in (66) is zero. Therefore the control (62) is optimal. In addition, by Lemma 2 we can write

$$
\begin{aligned}
h(\mathbf{v},\mathbf{B}) &\leq \frac{2\beta}{R} m(\mathbf{v},\mathbf{B}) - \beta \int_0^d v_{wall}^2 dx - \beta \int_0^d \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right]^2 dx - 2\beta g_4 \int_0^d (b^v_{top\_wall})^2 dx \\
&\quad -2\beta g_5 \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2 + 2\beta \Bigg(g_1 m(\mathbf{v},\mathbf{B}) + g_2 m^2(\mathbf{v},\mathbf{B}) + g_3 \int_0^d v_{wall}^2 dx \\
&\quad + g_4 \int_0^d (b^v_{top\_wall})^2 dx + g_5 \left(\int_0^d (b^v_{top\_wall})^2 dx \right)^2 \Bigg) \\
&\leq l_1 m(\mathbf{v},\mathbf{B}) + l_2 m^2(\mathbf{v},\mathbf{B}) - l_3 \int_0^d v_{wall}^2 dx - \beta \int_0^d \left[k_1 \Delta p + k_2 \frac{\Delta \left(b^{v^2} \right)}{2} \right]^2 dx.
\end{aligned}
$$

□

Inequality (63) implies that $h(\mathbf{v},\mathbf{B})$ cannot be made large without making $m(\mathbf{v},\mathbf{B})$ large as long as β is chosen to make $k_3 > 0$. Thus, the control law (62) maximizes mixing, with minimal control (v_{wall}) and sensing $\left(\Delta p, \Delta \left(b^{v^2}\right)\right)$ effort.

It is interesting to note that the control law (62) is independent of the physical parameters of the flow and therefore completely robust against their uncertainties. The proposed controller requires direct measurement of physical variables (pressure and induced magnetic field) only at the boundary of the flow domain. By pairing up the collocated pressure and magnetic field sensors, and velocity actuators, the need for centralized computation and wiring is eliminated. These important properties make the proposed control strategy implementation-ready.

VI. Numerical Method

A direct numerical simulation is performed based on the full MHD equations (Navier-Stokes equation and Magnetic Induction equation). A code based on the full MHD equations is necessary in this case to allow the measurement of the induced magnetic field at the boundary, as is required by the proposed control law (62). Although it is possible to find past and present research on the simulation of the full MHD equations for compressible flows, the work done on unsteady incompressible flows is scarce. The implementation of a code for simulating the full MHD equations in an incompressible flow is a very challenging problem. Firstly, one source of difficulty resides on the multiple time scales of the system; while the momentum equation is characterized by a $R \gg 1$, the induction equation is characterized by a $R_m \ll 1$. Secondly, the MHD equations become stiffer as the magnetic Reynolds number decreases. Generally, the compressible MHD equations, written in flux vector formulation, are numerically solved using artificial viscosity or nonlinear flux limiters. Finally, the numerically-computed magnetic and velocity fields must satisfy the incompressibility conditions. When the full set of MHD equations is solved numerically, the experience has shown that it is very difficult for the magnetic field to remain divergence-free. Numerous techniques have been proposed to deal with these numerical problems.[21–30] Based on the similar structures of the Navier-Stokes and Magnetic Induction equations, our first approach to the problem was to integrate the equations with different integration steps on a staggered grid within a periodic channel flow geometry using a hybrid Fourier pseudospectral–finite difference discretization and the fractional step technique. Taking advantage of the periodic boundary conditions in the streamwise (x) direction, this direction is discretized using Fourier pseudospectral methods,[31] while the wall-normal (y) direction is discretized using central finite differences on a non-uniform staggered grid.[32] The equations are integrated in time using a fractional step method,[33] designed to ensure the fulfillment of the divergence-free conditions, based on a hybrid Runge-Kutta/Crank-Nicolson time discretization.[34] The nonlinear terms are integrated explicitly using a fourth-order, low-storage Runge-Kutta method, while the viscous terms are treated implicitly using the Crank-Nicolson method.

VII. Simulation Results

In this section, the laminarization property of the imposed magnetic field is studied numerically. In addition, the effectiveness of the proposed control law (62) for mixing enhancement is numerically tested. All the simulation studies are carried out for the same flow domain:

$$\begin{aligned} -1 <& y < 1 \\ 0 <& x < 4\pi \end{aligned}$$

The same mesh is used in all the simulations presented in this section (NX = 150 andNY = 128). Simulation studies are carried out for different Reynolds numbers following a specific procedure: first, a fully established hydrodynamic flow is calculated assuming that there is no magnetic field present in the system; second, a magnetic field is imposed on the fully hydrodynamic established flow, which leads to another fully established magnetohydrodynamic flow with lower perturbation energy or even to a linear stable magnetohydrodynamic flow; finally, boundary feedback control is applied to the magnetohydrodynamic flow and an increase of the complexity of the flow pattern is observed. This suggests improved mixing, which is confirmed by studies of the behavior of dye blobs positioned in the flow.

A. Hydrodynamic Channel Flow

Incompressible conventional flows in 2D channels can be stable for low Reynolds numbers, as infinitesimal perturbations in the flow field are damped out. The flows turn unstable for high Reynolds numbers (Re > 5772).[35] Such flows usually reach a statistically steady state, which we call *fully established flows.* The full MHD code is capable of simulating 2D pure hydrodynamic channel flows by simply setting $B_0 = 0$, which means that no magnetic field is imposed. Three pure hydrodynamics flows are simulated: Re = 6000 (Figure 5(a)), Re = 7500 (Figure 6(a)), and Re = 10000 (Figure 7(a)). The flux mass Q is fixed to 1.5. In all three cases, a fully established flow showing some degree of vorticity is reached when the initial equilibrium velocity profile is infinitesimally perturbed at $t = 0$. The initial velocity profile is the parabolic equilibrium solution of the Navier-Stokes equation, which is linearly unstable for these Reynolds numbers.

B. Stabilization Effect of the Imposed Magnetic Field in MHD Flows

Simulation studies show the stabilizing effect of the imposed magnetic field in the 2D Hartmann flow. These simulation studies start at $t = 0$ with the fully established flows reached in the pure hydrodynamic channels (Subsection A). Magnetic fields of three different levels of strength ($\mathrm{B}_0^* = 0.1, 0.2, 0.3$) are imposed on each fully established flow at time $t = 0$. The magnetic Reynolds number is $\mathrm{Re_m} = 0.1$ in all cases. Observing the vorticity maps, it is interesting to note that weak magnetic fields ($\mathrm{Ha} < 3$) have significant stabilization effects on the fully established flows for all the Reynolds numbers: $\mathrm{Re} = 6000$ (Figure 5(b)), $\mathrm{Re} = 7500$ (Figure 6(b)), and $\mathrm{Re} = 10000$ (Figure 7(b)). Flows with lower Reynolds number, with stronger tendency towards stability, are more easily stabilized by the magnetic fields. The perturbation energy of the velocity field, defined as

$$E(\mathbf{v}) = \int_{-1}^{1} \int_{0}^{d} (u^2 + v^2) dx dy, \tag{67}$$

can be used to quantify the level of stability, or instability, of the flow. The evolutions in time of the perturbation energies are presented for the Reynolds numbers under consideration: $\mathrm{Re} = 6000$ (Figure 5(c)), $\mathrm{Re} = 7500$ (Figure 6(c)), and $\mathrm{Re} = 10000$ (Figure 7(c)). In all cases, the kinetic energy of the flow is reduced by the imposed magnetic field, and another fully established flow with lower perturbation energy, or even a linear stable flow, is reached.

C. Active Boundary Feedback Control of MHD flows

The effect of active boundary feedback control in each fully established MHD flow presented in Subsection B is studied here. The time evolution of the vorticity maps are shown for $\mathrm{B}_0^* = 0.3$ and the different considered Reynold numbers: $\mathrm{Re} = 6000$ (Figure 8(a)), $\mathrm{Re} = 7500$ (Figure 9(a)), and $\mathrm{Re} = 10000$ (Figure 10(a)). The stabilization effect of lower Reynolds number and stronger magnetic fields can be clearly seen from the plots. Figures 8(b) ($\mathrm{Re} = 6000$), 9(b) ($\mathrm{Re} = 7500$), and 10(b) ($\mathrm{Re} = 10000$) show the perturbation energy $E(v)$ and the control effort $C(v)$, which is defined as

$$C(v) = \sqrt{\int_{0}^{d} v(x, -1, t)^2 + v(x, 1, t)^2 dx}. \tag{68}$$

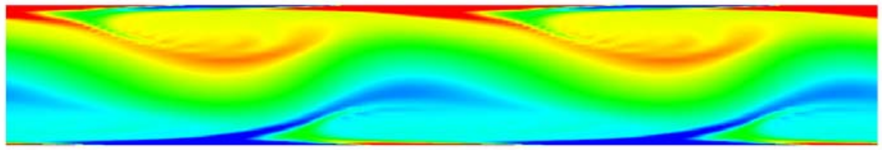
The ratio between the kinetic energy of the boundary control flow and the perturbation kinetic energy, $C^2(v)/E(v)$, is less than 1%, which suggests that small control can result in considerable mixing effect. A intuitive representation of the control mechanism can be seen from the boundary zoom-in (Figure 8(c)). The detailed velocity vectors show that the boundary control is pushing, by blowing, the nearby vortex into the center of the flow.

A particle tracking analysis is carried out to further visualize the mixing effectiveness of the control law. The tracking starts with the particles concentrated on several circular regions (Figure 11). The particles are assumed to exactly follow the fluid motion. Figures 12, 13, and 14 show the particle map evolutions for the controlled flows associated with Reynold numbers $\mathrm{Re} = 6000$, $\mathrm{Re} = 7500$, and $\mathrm{Re} = 10000$ respectively.

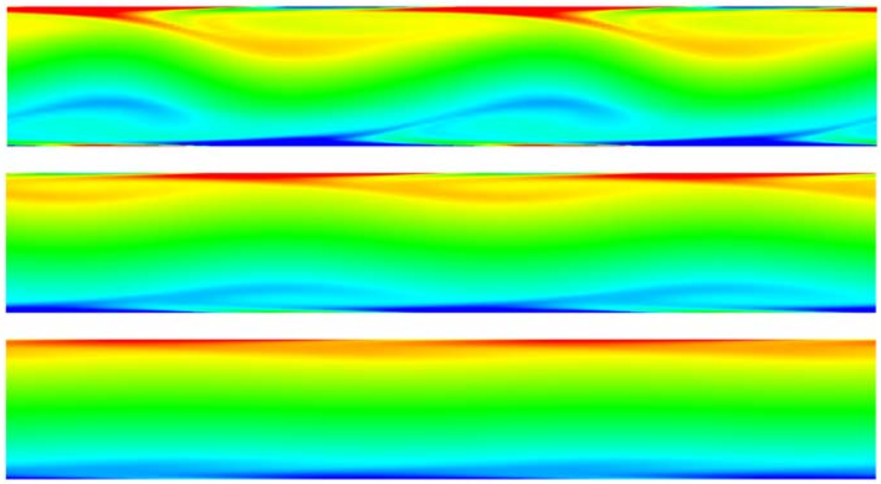
VIII. Conclusions and Future Work

Using the L_2-norm of first-order spatial derivatives of the velocity and magnetic field perturbations as a measure of mixing, a nonlinear Lyapunov-based boundary feedback control law that maximizes this measure, minimizing the control and sensing efforts, was designed for a 2D Hartman flow. The effectiveness of the optimal controller in enhancing mixing in 2D Hartmann flow was demonstrated in direct numerical simulations using a full MHD code based on a hybrid Fourier pseudospectral-finite difference discretization scheme and the fractional step technique.

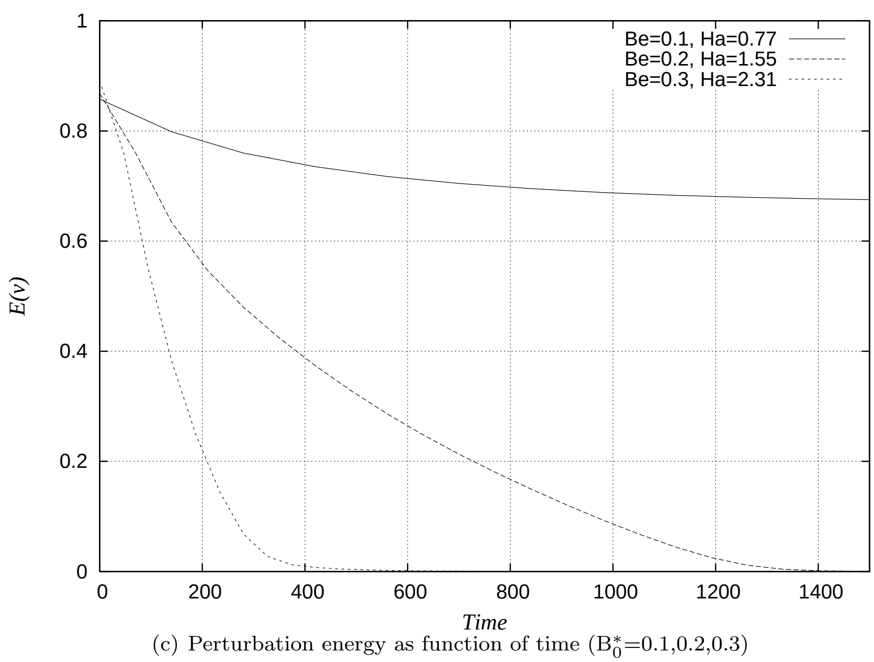
In conventional flows, although the nature of turbulence in the uncontrolled channel flow is inherently three dimensional, the control strategies for 2D and 3D channels do not differ. Therefore, focusing on the 2D case reduces the effort expended on numerical issues and allows to concentrate on control issues. In the case of the mixing problem, studying the 2D problem is indeed conservative: the neglected 3D instability mechanisms may be expected to substantially increase the rate of mixing beyond that seen in 2D model flows. This may not be the case for MHD flows, where the phenomena are fundamentally 3D (the 2D model flow not only neglects 3D turbulence (destabilizing) mechanisms but also 3D Lorentz force (stabilizing) mechanisms). For this reason, attention will be given in the future to "general" 2D, where the flow variables are functions of two space variables but have three components, and 3D geometries. In addition, the heat equation will be incorporated into the code in order to assess the effect of feedback boundary control on the heat transfer properties of the system.



(a) Fully established flow (t=0)

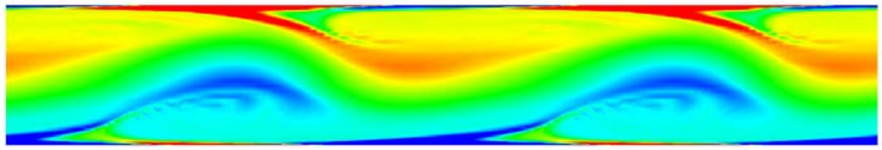


(b) Flow being stablized by magnetic field ($B_0^* = 0.3$, t=140,260,374)

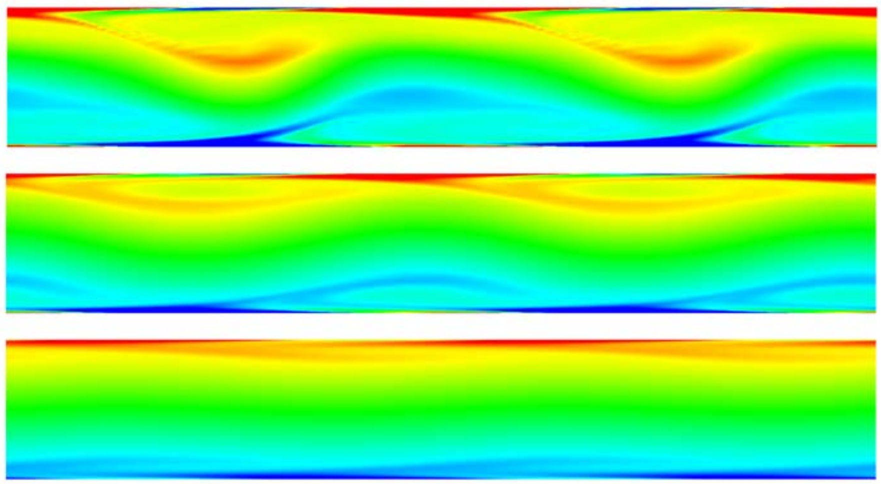


(c) Perturbation energy as function of time (B_0^*=0.1,0.2,0.3)

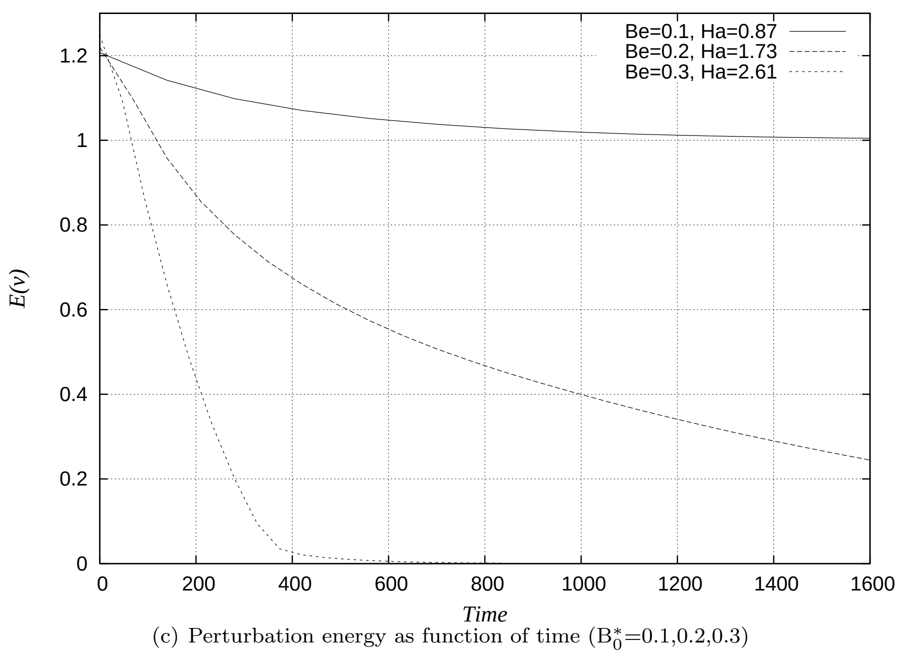
Figure 5. Simulation results at Re=6000



(a) Fully established flow (t=0)

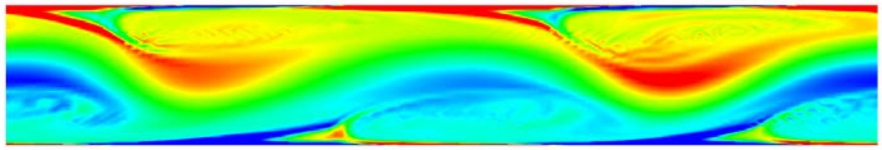


(b) Flow being stablized by magnetic field ($B_0^* = 0.3$, t=140,285,374)

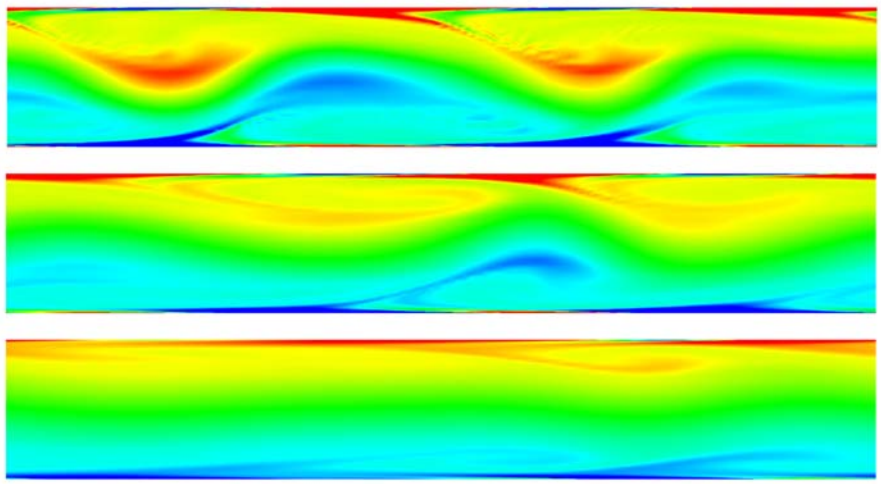


(c) Perturbation energy as function of time (B_0^*=0.1,0.2,0.3)

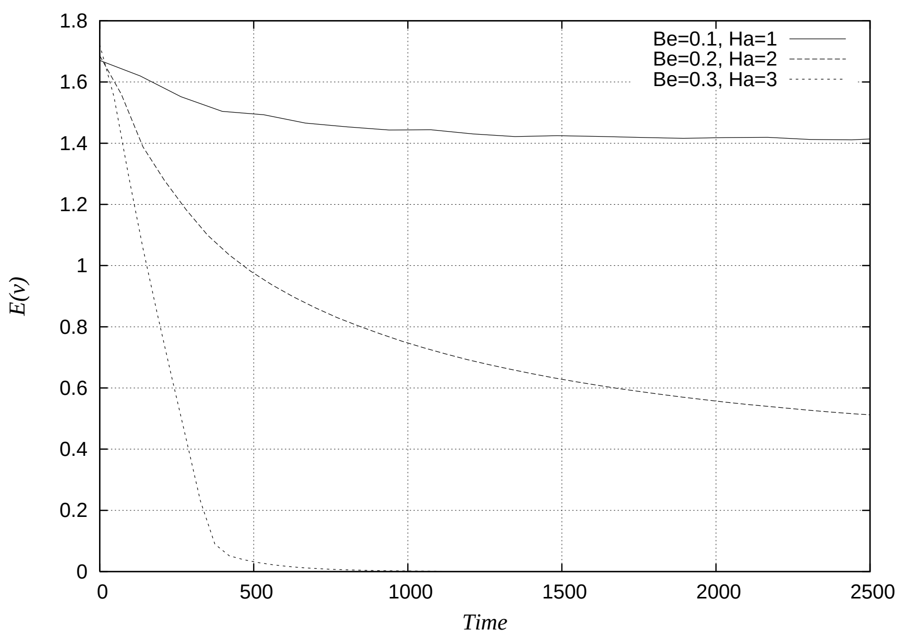
Figure 6. Simulation results at Re=7500



(a) Fully established flow (t=0)

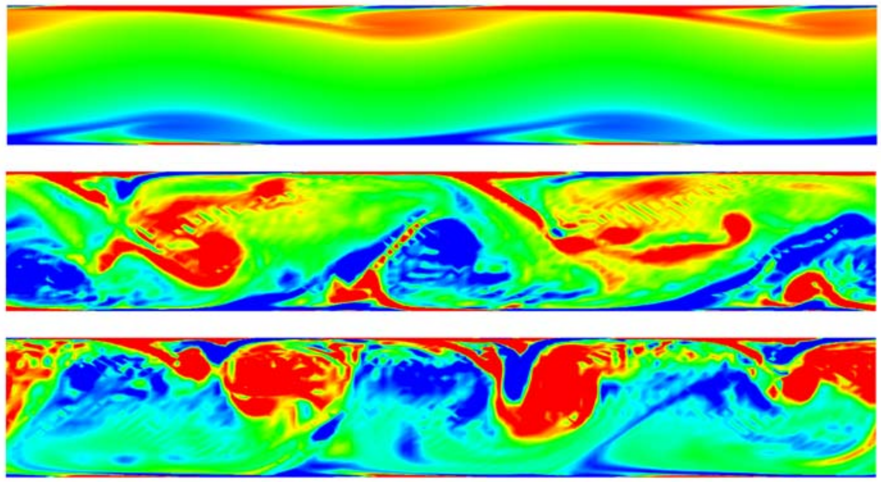


(b) Flow being stablized by magnetic field ($B_0^* = 0.3$, t=140,281,374)

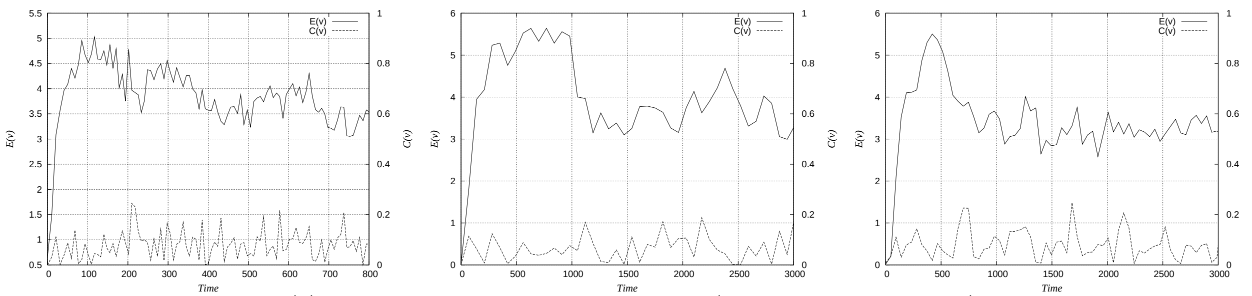


(c) Perturbation energy as function of time (B_0^*=0.1,0.2,0.3)

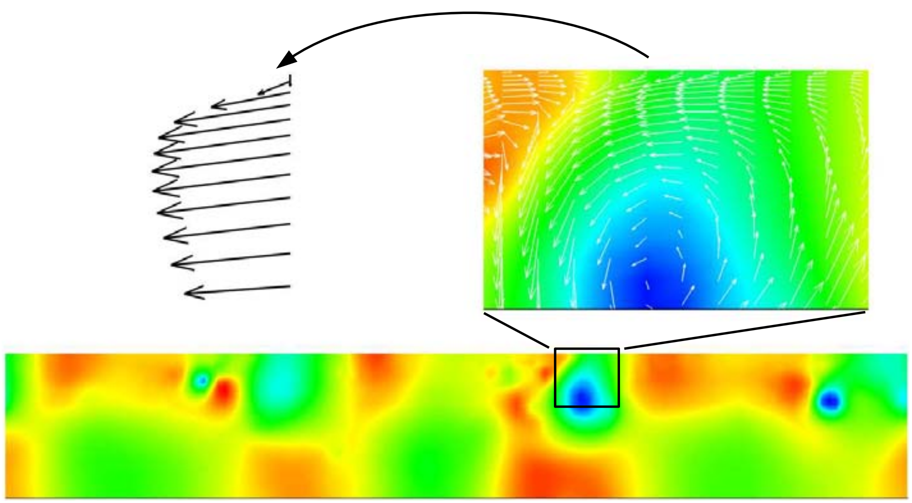
Figure 7. Simulation results at Re=10000



(a) Flow being destabilized by boundary control (Re=6000, $B_0^* = 0.3$, t=47,187,654)

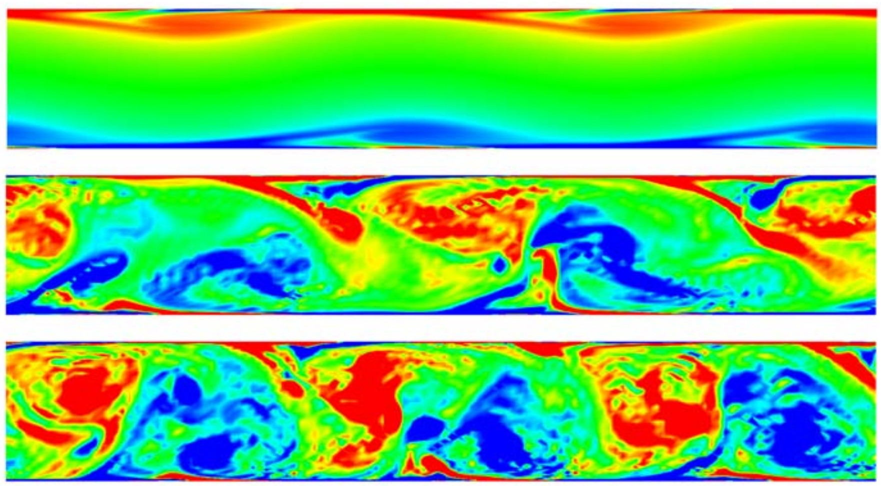


(b) Perturbation energy and control effort (Re=6000, B_0^*=0.1,0.2,0.3)

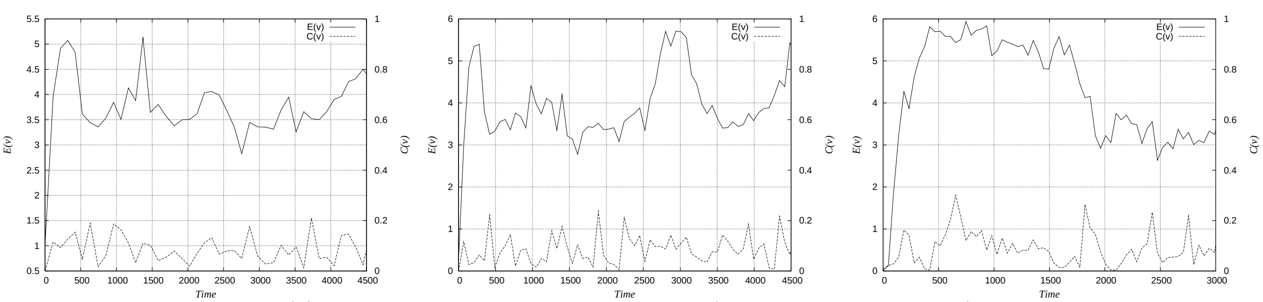


(c) Pressure and velocity zoom at boundary (Re=6000, $B_0^* = 0.3$)

Figure 8. Controlled flow at Re=6000

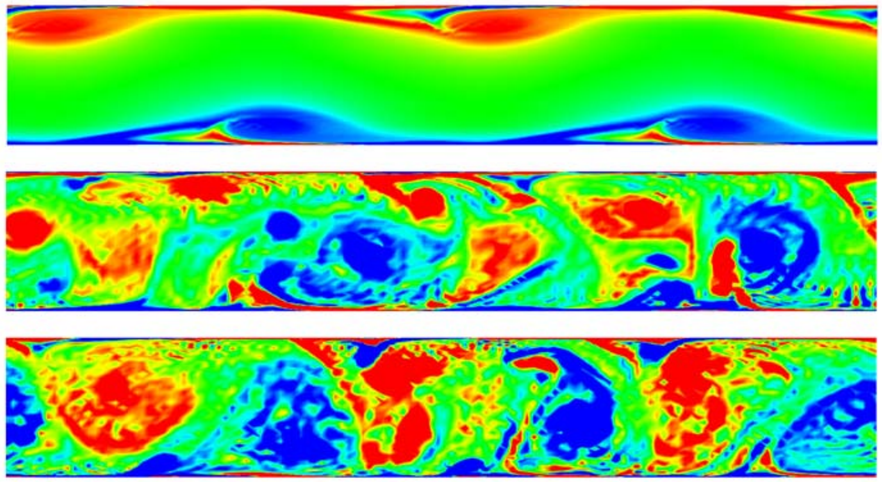


(a) Flow being destablized by boundary control (Re=7500, $B_0^* = 0.3$, t=47,140,327)

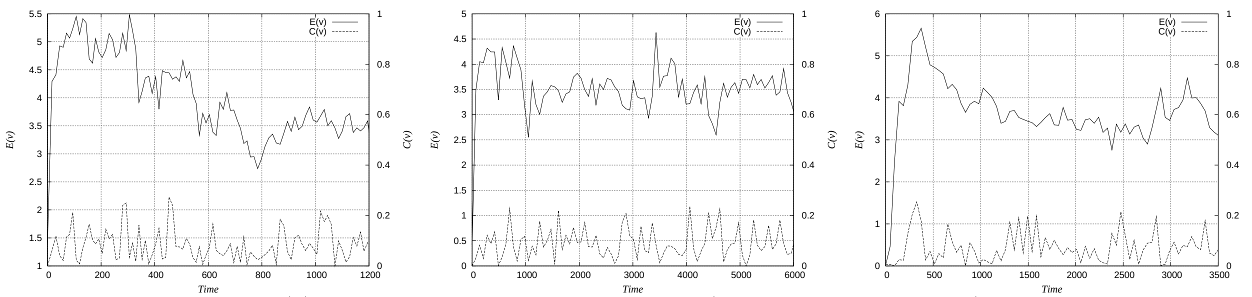


(b) Perturbation energy and control effort (Re=7500, B_0^*=0.1,0.2,0.3)

Figure 9. Controlled flow at Re=7500



(a) Flow being destabilized by boundary control (Re=10000, $B_0^* = 0.3$, t=47,140,280)



(b) Perturbation energy and control effort (Re=10000, B_0^*=0.1,0.2,0.3)

Figure 10. Controlled flow at Re=10000

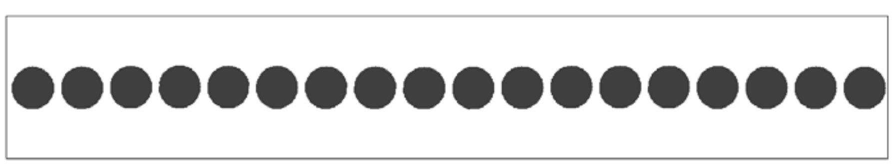


Figure 11. Initial particle distribution (t=0)

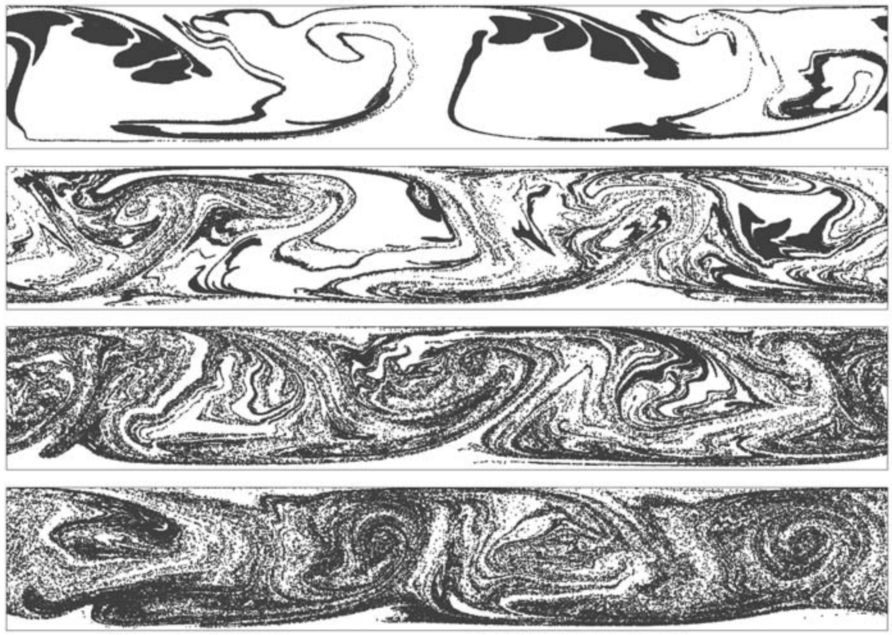


Figure 12. Particle distribution for controlled flow (Re=6000, $B_0^* = 0.3$**, t=47,187,654)**

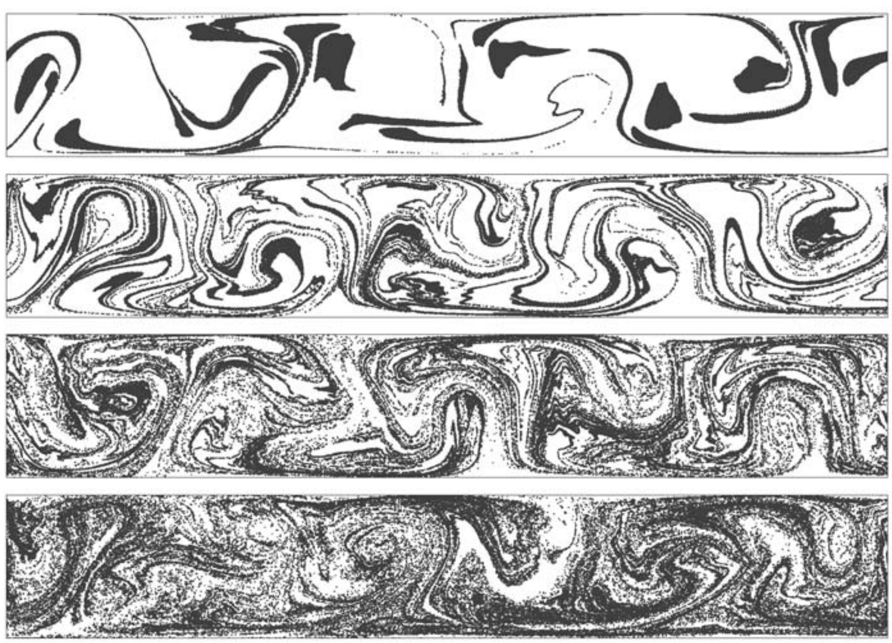


Figure 13. Particle distribution for controlled flow (Re=7500, $B_0^* = 0.3$**, t=47,140,327)**

Figure 14. Particle distribution for controlled flow (Re=10000, $B_0^* = 0.3$, t=47,140,280)

Appendix A: Induction Equation

Ohm's law:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{A-1}$$

Faraday's law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{A-2}$$

Ampere's law:

$$\mu \mathbf{j} = \nabla \times \mathbf{B} \tag{A-3}$$

B is solenoidal:

$$\nabla \cdot \mathbf{B} = 0 \tag{A-4}$$

v is solenoidal:

$$\nabla \cdot \mathbf{v} = 0 \tag{A-5}$$

From Ohm's law (A-1) we can write the electrical field **E** as

$$\mathbf{E} = \frac{\mathbf{j}}{\sigma} - \mathbf{v} \times \mathbf{B} \tag{A-6}$$

and rewrite the Faraday's law (A-2) as

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\frac{\mathbf{j}}{\sigma} - \mathbf{v} \times \mathbf{B}). \tag{A-7}$$

Considering, in addition, Ampere's law (A-3), it is possible to write the current density **j** as

$$\mathbf{j} = \frac{\nabla \times \mathbf{B}}{\mu} \tag{A-8}$$

and obtain

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\frac{1}{\sigma}\nabla \times \left(\frac{\nabla \times \mathbf{B}}{\mu}\right) + \nabla \times (\mathbf{v} \times \mathbf{B}) && \text{(A-9)} \\ &= -\frac{1}{\mu\sigma}\left[\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}\right] + \left[(\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{B} + \mathbf{v}(\nabla \cdot \mathbf{B})\right] && \text{(A-10)} \end{aligned}$$

where we have used the vectorial relationships

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{a}) &= \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \\ \nabla \times (\mathbf{a} \times \mathbf{b}) &= (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}).\end{aligned}$$

Taking into account from equations (A-4) and (A-5) that $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{v} = 0$ we can finally write

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{B} = \frac{1}{\mu\sigma}\nabla^2 \mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{v}. \tag{A-11}$$

Appendix B: Non-dimensional Equations

Let us start with the transport equation of linear momentum

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] = -\nabla P + \rho\nu\nabla^2 \mathbf{v} + \mathbf{f} + \mathbf{j} \times \mathbf{B}, \tag{B-1}$$

defining the dimensionless variables

$$v^* = \frac{v}{v_o} \tag{B-2}$$

$$\mathbf{x}^* = \frac{\mathbf{x}}{L} \tag{B-3}$$

$$t^* = \frac{v_o t}{L} \tag{B-4}$$

$$\mathbf{B}^* = \frac{\mathbf{B}}{b_o} \tag{B-5}$$

$$\mathbf{j}^* = \frac{j}{\sigma v_o b_o} \tag{B-6}$$

$$\mathbf{f}^* = \frac{L\mathbf{f}}{\rho v_o^2} \tag{B-7}$$

we can write

$$\frac{\partial \mathbf{v}^*}{\partial t^*} + (\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^* = -\nabla^* P^* + \frac{1}{R}\nabla^{*^2}\mathbf{v}^* + \mathbf{f}^* + N\left(\mathbf{j}^* \times \mathbf{B}^*\right), \tag{B-8}$$

where $R = \frac{v_o L}{\nu}$ is the Reynolds number, and $N = \frac{\sigma L b_o^2}{\rho v_o}$ is the Stuart number. In addition, we can write

$$\mathbf{j}^* = \frac{1}{R_m}\left(\nabla^* \times \mathbf{B}^*\right), \tag{B-9}$$

where $R_m = \mu\sigma v_o L$ is the magnetic Reynolds number, and rewrite the non-dimensional momentum equation as

$$\frac{\partial \mathbf{v}^*}{\partial t^*} + (\mathbf{v}^* \cdot \nabla^*)\mathbf{v}^* = -\nabla^* P^* + \frac{1}{R}\nabla^{*^2}\mathbf{v}^* + \mathbf{f}^* + \frac{N}{R_m}\left[(\nabla^* \times \mathbf{B}^*) \times \mathbf{B}^*\right]. \tag{B-10}$$

The $\hat{\mathbf{x}}$ component of equation (B-10) can be written as

$$\frac{\partial U^*}{\partial t^*} + (U^*\frac{\partial}{\partial x^*} + V^*\frac{\partial}{\partial y^*})U^* = -\frac{\partial P^*}{\partial x^*} + \frac{1}{R}\left(\frac{\partial^2 U^*}{\partial x^{*^2}} + \frac{\partial^2 U^*}{\partial y^{*^2}}\right) - \frac{N}{R_m}B^{v^*}\left(\frac{\partial B^{v^*}}{\partial x^*} - \frac{\partial B^{u^*}}{\partial y^*}\right),$$

which in steady state reduces to

$$\begin{aligned} 0 &= -\frac{\partial \bar{P}^*}{\partial x^*} + \frac{1}{R}\frac{\partial^2 \bar{U}^*}{\partial y^{*^2}} + \frac{N}{R_m}B_o^*\frac{\partial \bar{b}^*}{\partial y^*} \\ 0 &= -\frac{\partial \bar{P}^*}{\partial x^*} + \frac{\nu}{v_o L}\frac{\partial^2 \bar{U}^*}{\partial y^{*^2}} + \frac{b_o^2}{\mu\rho v_o^2}B_o^*\frac{\partial \bar{b}^*}{\partial y^*} \\ -1 &= \frac{\nu}{v_o L\left(-\frac{\partial \bar{P}^*}{\partial x^*}\right)}\frac{\partial^2 \bar{U}^*}{\partial y^{*^2}} + \frac{b_o^2}{\mu\rho v_o^2\left(-\frac{\partial \bar{P}^*}{\partial x^*}\right)}B_o^*\frac{\partial \bar{b}^*}{\partial y^*}, \end{aligned}$$

since $U^*(x^*,y^*,t^*) = \bar{U}^*(y^*)$, $V^*(x^*,y^*,t^*) = \bar{V}^*(y^*) = 0$, $B^{u^*}(x^*,y^*,t^*) = \bar{B}^{u^*}(y^*) = \bar{b}^*(y^*)$ and $B^{v^*}(x^*,y^*,t^*) = \bar{B}^{v^*}(y) = B_o^*$. Recalling that

$$\begin{aligned} \frac{\partial \bar{P}^*}{\partial x^*} &= \frac{L}{\rho v_o^2}\frac{\partial \bar{P}}{\partial x} \\ B_o^* &= \frac{B_o}{b_o} \end{aligned}$$

we can write

$$-1 = \frac{\nu \rho v_o}{L^2\left(-\frac{\partial \bar{P}}{\partial x}\right)}\frac{\partial^2 \bar{U}^*}{\partial {y^*}^2} + \frac{b_o B_o}{\mu L\left(-\frac{\partial \bar{P}}{\partial x}\right)}\frac{\partial \bar{b}^*}{\partial y^*}.$$

Making

$$\begin{aligned} v_o &= \frac{L^2\left(-\frac{\partial \bar{P}}{\partial x}\right)}{\rho \nu} \\ b_o &= \mu L^2\sqrt{\frac{\sigma}{\rho\nu}}\left(-\frac{\partial \bar{P}}{\partial x}\right) \end{aligned}$$

we can recover equation (17)

$$-1 = \frac{\partial^2 \bar{U}^*}{\partial {y^*}^2} + Ha\frac{\partial \bar{b}^*}{\partial y^*} \qquad \text{(B-11)}$$

where $Ha = B_o L\sqrt{\frac{\sigma}{\rho\nu}}$.

Let us now consider the equation of magnetic induction

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B} = \frac{1}{\mu\sigma}\nabla^2\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{v}. \qquad \text{(B-12)}$$

Using the dimensionless variables v^*, x^*, t^*, and B^*, we can write the non-dimensional induction equation

$$\frac{\partial \mathbf{B}^*}{\partial t^*} + (\mathbf{v}^*\cdot\nabla^*)\mathbf{B}^* = \frac{1}{R_m}{\nabla^*}^2\mathbf{B}^* + (\mathbf{B}^*\cdot\nabla^*)\mathbf{v}^*. \qquad \text{(B-13)}$$

The $\hat{\mathbf{x}}$ component of equation (B-13) can be written as

$$\frac{\partial B^{u^*}}{\partial t^*} + (U^*\frac{\partial}{\partial x^*} + V^*\frac{\partial}{\partial y^*})B^{u^*} = \frac{1}{R_m}\left(\frac{\partial^2 B^{u^*}}{\partial {x^*}^2} + \frac{\partial^2 B^{u^*}}{\partial {y^*}^2}\right) + \left(B^{u^*}\frac{\partial}{\partial x^*} + B^{v^*}\frac{\partial}{\partial y^*}\right)U^*,$$

which in steady state reduces to

$$\begin{aligned} 0 &= \frac{1}{R_m}\frac{\partial^2 \bar{b}^*}{\partial {y^*}^2} + B_o^*\frac{\partial \bar{U}^*}{\partial y^*} \\ 0 &= \frac{\partial^2 \bar{b}^*}{\partial {y^*}^2} + \mu\sigma L v_o B_o^*\frac{\partial \bar{U}^*}{\partial y^*}. \end{aligned}$$

Recalling that

$$B_o^* = \frac{B_o}{b_o}$$

we can write

$$0 = \frac{\partial^2 \bar{b}^*}{\partial {y^*}^2} + \frac{\mu\sigma L v_o B_o}{b_o}\frac{\partial \bar{U}^*}{\partial y^*}$$

and recover equation (16)

$$0 = \frac{\partial^2 \bar{b}^*}{\partial {y^*}^2} + Ha\frac{\partial \bar{U}^*}{\partial y^*}.$$

Appendix C: Proof of Lemma 1

Integrating by parts expression (30) we have

$$
\begin{aligned}
& \int_{-1}^{1}\int_{0}^{d} -u\left[\bar{U}u_x + uu_x + v\bar{U}' + vu_y - \frac{1}{R}\left(u_{xx}+u_{yy}\right) + p_x\right]dxdy \\
= \; & \int_{-1}^{1}\int_{0}^{d}\left[-\bar{U}uu_x - u^2u_x - uv\bar{U}' - vuu_y + \frac{1}{R}u\left(u_{xx}+u_{yy}\right) - up_x\right]dxdy \\
= \; & -\int_{-1}^{1}\int_{0}^{d}\bar{U}\frac{1}{2}(u^2)_x dxdy - \int_{-1}^{1}\int_{0}^{d}\bar{U}'uvdxdy - \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}u(u^2)_x dxdy \\
& -\int_{-1}^{1}\int_{0}^{d}\frac{1}{2}v(u^2)_y dxdy + \int_{-1}^{1}\int_{0}^{d}\frac{1}{R}uu_{xx}dxdy + \int_{-1}^{1}\int_{0}^{d}\frac{1}{R}uu_{yy}dxdy - \int_{-1}^{1}\int_{0}^{d}up_x dxdy \\
= \; & -\underbrace{\int_{-1}^{1}\bar{U}\frac{1}{2}u^2\,\big|_0^d dy}_{=0} - \int_{-1}^{1}\int_{0}^{d}\bar{U}'uvdxdy - \underbrace{\int_{-1}^{1}\frac{1}{2}uu^2\,\big|_0^d dy}_{=0} + \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}u^2u_x dxdy \\
& -\underbrace{\int_{0}^{d}\frac{1}{2}vu^2\,\big|_{-1}^{1}dx}_{=0} + \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}u^2v_y dxdy + \underbrace{\frac{1}{R}\int_{-1}^{1}uu_x\,\big|_0^d dy}_{=0} - \frac{1}{R}\int_{-1}^{1}\int_{0}^{d}(u_x)^2 dxdy \\
& +\underbrace{\frac{1}{R}\int_{0}^{d}uu_y\,\big|_{-1}^{1}dx}_{=0} - \frac{1}{R}\int_{-1}^{1}\int_{0}^{d}(u_y)^2 dxdy - \underbrace{\int_{-1}^{1}up\,\big|_0^d dy}_{=0} + \int_{-1}^{1}\int_{0}^{d}u_x p dxdy \\
= \; & -\int_{-1}^{1}\int_{0}^{d}\bar{U}'uvdxdy + \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}u^2\underbrace{(u_x+v_y)}_{=0\ (25)}dxdy - \frac{1}{R}\int_{-1}^{1}\int_{0}^{d}((u_x)^2+(u_y)^2)dxdy \\
& +\int_{-1}^{1}\int_{0}^{d}u_x p dxdy \\
= \; & -\int_{-1}^{1}\int_{0}^{d}\bar{U}'uvdxdy - \frac{1}{R}\int_{-1}^{1}\int_{0}^{d}((u_x)^2+(u_y)^2)dxdy + \int_{-1}^{1}\int_{0}^{d}u_x p dxdy
\end{aligned}
\tag{C-1}
$$

Integrating by parts expression (32) we have

$$
\begin{aligned}
& \int_{-1}^{1}\int_{0}^{d} -v\left[\bar{U}v_x + uv_x + vv_y - \frac{1}{R}\left(v_{xx}+v_{yy}\right)+p_y\right]dxdy \\
= \; & \int_{-1}^{1}\int_{0}^{d}\left[-\bar{U}vv_x - uvv_x - v^2v_y + \frac{1}{R}v\left(v_{xx}+v_{yy}\right) - vp_y\right]dxdy \\
= \; & -\int_{-1}^{1}\int_{0}^{d}\bar{U}\frac{1}{2}(v^2)_x dxdy - \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}u(v^2)_x dxdy - \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}v(v^2)_y dxdy \\
& + \int_{-1}^{1}\int_{0}^{d}\frac{1}{R}vv_{xx}dxdy + \int_{-1}^{1}\int_{0}^{d}\frac{1}{R}vv_{yy}dxdy - \int_{-1}^{1}\int_{0}^{d}vp_y dxdy \\
= \; & -\underbrace{\int_{-1}^{1}\bar{U}\frac{1}{2}v^2\,\Big|_0^d dy}_{=0} - \underbrace{\int_{-1}^{1}\frac{1}{2}uv^2\,\Big|_0^d dy}_{=0} + \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}v^2u_x dxdy \\
& -\underbrace{\int_{0}^{d}\frac{1}{2}vv^2\,\Big|_{-1}^{1}dx}_{=0} + \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}v^2v_y dxdy + \frac{1}{R}\underbrace{\int_{-1}^{1}vv_x\,\Big|_0^d dy}_{=0} - \frac{1}{R}\int_{-1}^{1}\int_{0}^{d}(v_x)^2 dxdy \\
& \frac{1}{R}\int_{0}^{d}vv_y\,\Big|_{-1}^{1}dx - \frac{1}{R}\int_{-1}^{1}\int_{0}^{d}(v_y)^2dxdy - \int_{0}^{d}vp\,\Big|_{-1}^{1}dx + \int_{-1}^{1}\int_{0}^{d}v_y p dxdy \\
= \; & \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}v^2\underbrace{(u_x+v_y)}_{=0\ (25)}dxdy - \frac{1}{R}\int_{-1}^{1}\int_{0}^{d}((v_x)^2+(v_y)^2)dxdy + \int_{0}^{d}v(\frac{v_y}{R}-p)\,\Big|_{-1}^{1}dx \\
& + \int_{-1}^{1}\int_{0}^{d}v_y p dxdy \\
= \; & -\frac{1}{R}\int_{-1}^{1}\int_{0}^{d}((v_x)^2+(v_y)^2)dxdy + \int_{-1}^{1}\int_{0}^{d}v_y p dxdy - \int_{0}^{d}vp\,\Big|_{-1}^{1}dx \\
= \; & -\frac{1}{R}\int_{-1}^{1}\int_{0}^{d}((v_x)^2+(v_y)^2)dxdy + \int_{-1}^{1}\int_{0}^{d}v_y p dxdy - \int_{0}^{d}v_{wall}\Delta p dx
\end{aligned}
\tag{C-2}
$$

where we have taken into account that $0 = u_x + v_y\,|_{y=\pm1} = v_y\,|_{y=\pm1}$ $(u\,|_{y=\pm1} = 0 \Rightarrow u_x\,|_{y=\pm1} = 0)$.

Integrating by parts expression (34) we have

$$\begin{aligned}
&\int_{-1}^{1}\int_{0}^{d} -b^u\left[\bar{U}b^u_x + ub^u_x + v\bar{b}^{'} + vb^u_y - \frac{1}{R_m}\left(b^u_{xx}+b^u_{yy}\right)\right]dxdy \\
=\quad & \int_{-1}^{1}\int_{0}^{d}\left[-\bar{U}b^u b^u_x - ub^u b^u_x - v\bar{b}^{'}b^u - vb^u b^u_y + \frac{b^u}{R_m}\left(b^u_{xx}+b^u_{yy}\right)\right]dxdy \\
=\quad & \int_{-1}^{1}\int_{0}^{d} -\bar{U}\frac{1}{2}(b^{u^2})_x dxdy - \int_{-1}^{1}\int_{0}^{d} u\frac{1}{2}(b^{u^2})_x dxdy - \int_{-1}^{1}\int_{0}^{d}\bar{b}^{'}vb^u dxdy \\
& -\int_{-1}^{1}\int_{0}^{d} v\frac{1}{2}(b^{u^2})_y dxdy + \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} b^u b^u_{xx}dxdy + \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} b^u b^u_{yy}dxdy \\
=\quad & -\underbrace{\int_{-1}^{1}\bar{U}\frac{1}{2}b^{u^2}\,|_0^d dy}_{=0} - \underbrace{\int_{-1}^{1}u\frac{1}{2}b^{u^2}\,|_0^d dy}_{=0} + \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}b^{u^2}u_x dxdy - \int_{-1}^{1}\int_{0}^{d}\bar{b}^{'}vb^u dxdy \\
& -\underbrace{\int_{0}^{d} v\frac{1}{2}b^{u^2}\,|_{-1}^{1}dx}_{=0} + \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}b^{u^2}v_y dxdy + \frac{1}{R_m}\underbrace{\int_{-1}^{1} b^u b^u_x\,|_0^d dy}_{=0} - \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d}(b^u_x)^2 dxdy \\
& +\frac{1}{R_m}\underbrace{\int_{-1}^{1}\int_{0}^{d} b^u b^u_y\,|_{-1}^{1}dx}_{=0} - \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d}(b^u_y)^2 dxdy \\
=\quad & \int_{-1}^{1}\int_{0}^{d}\frac{1}{2}b^{u^2}\underbrace{(u_x+v_y)}_{=0\ (25)}dxdy - \int_{-1}^{1}\int_{0}^{d}\bar{b}^{'}vb^u dxdy - \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d}\left[(b^u_x)^2+(b^u_y)^2\right]dxdy \\
=\quad & -\int_{-1}^{1}\int_{0}^{d}\bar{b}^{'}vb^u dxdy - \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d}\left[(b^u_x)^2+(b^u_y)^2\right]dxdy \qquad \text{(C-3)}
\end{aligned}$$

Integrating by parts expression (36) we have

$$
\begin{aligned}
& \int_{-1}^{1}\int_{0}^{d} -b^v \left[\bar{U} b_x^v + u b_x^v + v b_y^v - \frac{1}{R_m}\left(b_{xx}^v + b_{yy}^v\right)\right] dxdy \\
= & \int_{-1}^{1}\int_{0}^{d} \left(-\bar{U} b^v b_x^v - u b^v b_x^v - v b^v b_y^v + \frac{1}{R_m} b^v \left(b_{xx}^v + b_{yy}^v\right)\right) dxdy \\
= & -\int_{-1}^{1}\int_{0}^{d} \bar{U}\frac{1}{2}(b^{v^2})_x dxdy - \int_{-1}^{1}\int_{0}^{d} u\frac{1}{2}(b^{v^2})_x dxdy - \int_{-1}^{1}\int_{0}^{d} v\frac{1}{2}(b^{v^2})_y dxdy \\
& + \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} b^v b_{xx}^v dxdy + \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} b^v b_{yy}^v dxdy \\
= & \underbrace{-\int_{-1}^{1} \bar{U}\frac{1}{2} b^{v^2} |_0^d dy}_{=0} \underbrace{- \int_{-1}^{1} u\frac{1}{2} b^{v^2} |_0^d dy}_{=0} + \int_{-1}^{1}\int_{0}^{d} \frac{1}{2} b^{v^2} u_x dxdy - \int_{0}^{d} \frac{1}{2} v b^{v^2} |_{-1}^{1} dx \\
& + \int_{-1}^{1}\int_{0}^{d} \frac{1}{2} b^{v^2} v_y dxdy + \frac{1}{R_m}\underbrace{\int_{-1}^{1} b^v b_x^v |_0^d dy}_{=0} - \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} (b_x^v)^2 dxdy \\
& + \frac{1}{R_m}\int_{0}^{d} b^v b_y^v |_{-1}^{1} dx - \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} (b_y^v)^2 dxdy \\
= & \int_{-1}^{1}\int_{0}^{d} \frac{1}{2} b^{v^2} \underbrace{(u_x + v_y)}_{=0\ (25)} dxdy - \int_{0}^{d} \frac{1}{2} v b^{v^2} |_{-1}^{1} dx - \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} \left[(b_x^v)^2 + (b_y^v)^2\right] dxdy \\
& + \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} b^v b_y^v |_{-1}^{1} dx \\
= & -\int_{0}^{d} \frac{1}{2} v_{wall} \Delta\left(b^{v^2}\right) dx - \frac{1}{R_m}\int_{-1}^{1}\int_{0}^{d} \left[(b_x^v)^2 + (b_y^v)^2\right] dxdy
\end{aligned}
\tag{C-4}
$$

where we have taken into account that $0 = b_x^u + b_y^v |_{y=\pm 1} = b_y^v |_{y=\pm 1}$ $(b^u |_{y=\pm 1} = 0 \Rightarrow b_x^u |_{y=\pm 1} = 0)$. Adding (30), (32), (34) and (36), and taking into account (C-1), (C-2), (C-3) and (C-4) and (25) we obtain

$$
\begin{aligned}
& -\frac{k_1}{R}\int_{-1}^{1}\int_{0}^{d} (u_x^2 + u_y^2 + v_x^2 + v_y^2) dxdy - \frac{k_2}{R_m}\int_{-1}^{1}\int_{0}^{d} \left((b_x^u)^2 + (b_y^u)^2 + (b_x^v)^2 + (b_y^v)^2\right) dxdy \\
- & k_1 \int_{0}^{d} v_{wall}\Delta p dx - k_2 \int_{0}^{d} \frac{1}{2} v_{wall}\Delta\left(b^{v^2}\right) dx \\
- & k_1 \int_{-1}^{1}\int_{0}^{d} \bar{U}' uv dxdy - k_2 \int_{-1}^{1}\int_{0}^{d} \bar{b}' b^u v dxdy \\
= & -\frac{1}{R} m(\mathbf{v}, \mathbf{B}) - \int_{0}^{d} v_{wall}\left[k_1 \Delta p + k_2 \frac{\Delta\left(b^{v^2}\right)}{2}\right] dx - k_1 \int_{-1}^{1}\int_{0}^{d} \bar{U}' uv dxdy - k_2 \int_{-1}^{1}\int_{0}^{d} \bar{b}' b^u v dxdy
\end{aligned}
$$

Adding (31), (33), (35) and (37) we obtain

$$
\begin{aligned}
& \int_{-1}^{1}\int_{0}^{d} \frac{N}{R_m}\left\{\left(b_x^v - b_y^u\right)\left[(k_1\bar{b} + k_1 b^u)v - (k_1 B_o + k_1 b^v)u\right]\right\} dxdy \\
+ & \int_{-1}^{1}\int_{0}^{d} \left\{(\bar{b} + b^u)(k_2 b^u u_x + k_2 b^v v_x) + (B_o + b^v)(k_2 b^u u_y + k_2 b^v v_y)\right\} dxdy \\
+ & \int_{-1}^{1}\int_{0}^{d} \frac{N}{R_m}\bar{b}'(k_1 u b^v - k_1 v b^u) dxdy + \int_{-1}^{1}\int_{0}^{d} k_2 \bar{U}' b^u b^v dxdy
\end{aligned}
$$

Consequently we can write the time derivative of $E(\mathbf{v}, \mathbf{B})$ as

$$\dot{E}(\mathbf{v},\mathbf{B}) = -\frac{1}{R}m(\mathbf{v},\mathbf{B}) - \int_0^d v_{wall}\left[k_1\Delta p + k_2\frac{\Delta\left(b^{v^2}\right)}{2}\right]dx + g(\mathbf{v},\mathbf{B}). \qquad \square$$

Appendix D: Proof of Lemma 2

We can write

$$u(x,y,t) = u(x,1,t) - \int_y^1 u_y(x,y,t)dy = -\int_y^1 u_y(x,y,t)dy \tag{D-1}$$

$$v(x,y,t) = v(x,1,t) - \int_y^1 v_y(x,y,t)dy = v_{wall}(x,t) - \int_y^1 v_y(x,y,t)dy \tag{D-2}$$

$$b^u(x,y,t) = b^u(x,1,t) - \int_y^1 b^u_y(x,y,t)dy = -\int_y^1 b^u_y(x,y,t)dy \tag{D-3}$$

$$b^v(x,y,t) = b^v(x,1,t) - \int_y^1 b^v_y(x,y,t)dy = b^v_{top\_wall}(x,t) - \int_y^1 b^v_y(x,y,t)dy \tag{D-4}$$

and therefore

$$u^2(x,y,t) = \left(-\int_y^1 u_y dy\right)^2 \tag{D-5}$$

$$\begin{aligned}
v^2(x,y,t) &= \left(v_{wall}(x,t) - \int_y^1 v_y dy\right)^2 \\
&= v_{wall}^2 - 2v_{wall}\int_y^1 v_y dy + \left(\int_y^1 v_y dy\right)^2 \\
&\le v_{wall}^2 + 2bv_{wall}^2 + \frac{2}{b}\left(\int_y^1 v_y dy\right)^2 + \left(\int_y^1 v_y dy\right)^2 \\
&\le (1+2b)v_{wall}^2 + (1+\frac{2}{b})\left(\int_y^1 v_y dy\right)^2
\end{aligned} \tag{D-6}$$

$$(b^u)^2(x,y,t) = \left(-\int_y^1 b^u_y dy\right)^2 \tag{D-7}$$

$$\begin{aligned}
(b^v)^2(x,y,t) &= \left(b^v_{top_wall}(x,t) - \int_y^1 b^v_y dy\right)^2 \\
&= (b^v_{top_wall})^2 - 2b^v_{top_wall}\int_y^1 b^v_y dy + \left(\int_y^1 b^v_y dy\right)^2 \\
&\le (b^v_{top_wall})^2 + 2b(b^v_{top_wall})^2 + \frac{2}{b}\left(\int_y^1 b^v_y dy\right)^2 + \left(\int_y^1 b^v_y dy\right)^2 \\
&\le (1+2b)(b^v_{top_wall})^2 + (1+\frac{2}{b})\left(\int_y^1 b^v_y dy\right)^2
\end{aligned} \tag{D-8}$$

where we have used Young's inequality with $b > 0$ to write

$$\begin{aligned}
v_{wall}\int_y^1 v_y dy &\le bv_{wall}^2 + \frac{1}{b}\left(\int_y^1 v_y dy\right)^2 \\
b^v_{top_wall}\int_y^1 b^v_y dy &\le b(b^v_{top_wall})^2 + \frac{1}{b}\left(\int_y^1 b^v_y dy\right)^2.
\end{aligned}$$

By Schwartz inequality we can write

$$
\begin{aligned}
\left(\int_y^1 v_y dy\right)^2 = \left(\int_y^1 1v_y dy\right)^2 \le \left(\int_y^1 1^2 dy\right)\left(\int_y^1 v_y^2 dy\right) &= (1-y)\left(\int_y^1 v_y^2 dy\right) \\
&\le (1-y)\left(\int_{-1}^1 v_y^2 dy\right) \\
\left(\int_y^1 u_y dy\right)^2 = \left(\int_y^1 1u_y dy\right)^2 \le \left(\int_y^1 1^2 dy\right)\left(\int_y^1 u_y^2 dy\right) &= (1-y)\left(\int_y^1 u_y^2 dy\right) \\
&\le (1-y)\left(\int_{-1}^1 u_y^2 dy\right) \\
\left(\int_y^1 b_y^v dy\right)^2 = \left(\int_y^1 1b_y^v dy\right)^2 \le \left(\int_y^1 1^2 dy\right)\left(\int_y^1 (b_y^v)^2 dy\right) &= (1-y)\left(\int_y^1 (b_y^v)^2 dy\right) \\
&\le (1-y)\left(\int_{-1}^1 (b_y^v)^2 dy\right) \\
\left(\int_y^1 b_y^u dy\right)^2 = \left(\int_y^1 1b_y^u dy\right)^2 \le \left(\int_y^1 1^2 dy\right)\left(\int_y^1 (b_y^u)^2 dy\right) &= (1-y)\left(\int_y^1 (b_y^u)^2 dy\right) \\
&\le (1-y)\left(\int_{-1}^1 (b_y^u)^2 dy\right)
\end{aligned}
$$

and conclude that

$$
\begin{aligned}
\int_{-1}^1\int_0^d u^2 dxdy &\le \int_{-1}^1\int_0^d \left((1-y)\int_{-1}^1 u_y^2 dy\right) dxdy \\
&= \int_0^d \left(\int_{-1}^1 (1-y)dy\right)\left(\int_{-1}^1 u_y^2 dy\right) dx \\
&= 2\int_{-1}^1\int_0^d u_y^2 dxdy \qquad \text{(D-9)} \\
\int_{-1}^1\int_0^d v^2 dxdy &\le \int_{-1}^1\int_0^d (1+2b)v_{wall}^2 dxdy + \int_{-1}^1\int_0^d \left((1+\frac{2}{b})(1-y)\int_{-1}^1 v_y^2 dy\right) dxdy \\
&= 2(1+2b)\int_0^d v_{wall}^2 dx + (1+\frac{2}{b})\int_0^d \left(\int_{-1}^1 (1-y)dy\right)\left(\int_{-1}^1 v_y^2 dy\right) dx \\
&= 2(1+2b)\int_0^d v_{wall}^2 dx + 2(1+\frac{2}{b})\int_{-1}^1\int_0^d v_y^2 dxdy \qquad \text{(D-10)}
\end{aligned}
$$

$$
\begin{aligned}
\int_{-1}^1\int_0^d (b^u)^2 dxdy &\le \int_{-1}^1\int_0^d \left((1-y)\int_{-1}^1 (b_y^u)^2 dy\right) dxdy \\
&= \int_0^d \left(\int_{-1}^1 (1-y)dy\right)\left(\int_{-1}^1 (b_y^u)^2 dy\right) dx \\
&= 2\int_{-1}^1\int_0^d (b_y^u)^2 dxdy \qquad \text{(D-11)} \\
\int_{-1}^1\int_0^d (b^v)^2 dxdy &\le \int_{-1}^1\int_0^d (1+2b)(b_{top\_wall}^v)^2 dxdy + \int_{-1}^1\int_0^d \left((1+\frac{2}{b})(1-y)\int_{-1}^1 (b_y^v)^2 dy\right) dxdy \\
&= 2(1+2b)\int_0^d (b_{top\_wall}^v)^2 dx + (1+\frac{2}{b})\int_0^d \left(\int_{-1}^1 (1-y)dy\right)\left(\int_{-1}^1 (b_y^v)^2 dy\right) dx \\
&= 2(1+2b)\int_0^d (b_{top\_wall}^v)^2 dx + 2(1+\frac{2}{b})\int_{-1}^1\int_0^d (b_y^v)^2 dxdy \qquad \text{(D-12)}
\end{aligned}
$$

With this preliminary results we can start now finding bounds to each one of the terms (44)–(57) of $g(\mathbf{v},\mathbf{B})$. Equation (44) can be bounded as

$$-\int_{-1}^{1}\int_{0}^{d}\bar{U}^{'}uvdxdy \leq |\bar{U}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}|u||v|dxdy$$

and by Young's inequality

$$-\int_{-1}^{1}\int_{0}^{d}\bar{U}^{'}uvdxdy \leq \frac{1}{a_1}|\bar{U}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}u^2dxdy + a_1|\bar{U}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}v^2dxdy$$

and by equations (D-9) and (D-10)

$$\begin{aligned}-\int_{-1}^{1}\int_{0}^{d}\bar{U}^{'}uvdxdy &\leq 2a_1|\bar{U}^{'}|_{max}(1+2b_1)\int_{0}^{d}v_{wall}^2dx \\ &+|\bar{U}^{'}|_{max}\left\{\frac{2}{a_1}\int_{-1}^{1}\int_{0}^{d}u_y^2dxdy + 2a_1(1+\frac{2}{b_1})\int_{-1}^{1}\int_{0}^{d}v_y^2dxdy\right\}\end{aligned} \tag{D-13}$$

Equation (45) can be bounded as

$$-\int_{-1}^{1}\int_{0}^{d}\bar{b}^{'}b^uvdxdy \leq |\bar{b}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}|b^u||v|dxdy$$

and by Young's inequality

$$-\int_{-1}^{1}\int_{0}^{d}\bar{b}^{'}b^uvdxdy \leq \frac{1}{a_2}|\bar{b}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}(b^u)^2dxdy + a_2|\bar{b}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}v^2dxdy$$

and by equations (D-10) and (D-11)

$$\begin{aligned}-\int_{-1}^{1}\int_{0}^{d}\bar{b}^{'}b^uvdxdy &\leq 2a_2|\bar{b}^{'}|_{max}(1+2b_2)\int_{0}^{d}v_{wall}^2dx \\ &+|\bar{b}^{'}|_{max}\left\{\frac{2}{a_2}\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy + 2a_2(1+\frac{2}{b_2})\int_{-1}^{1}\int_{0}^{d}v_y^2dxdy\right\}\end{aligned} \tag{D-14}$$

Equation (46) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d}\bar{U}^{'}b^ub^vdxdy \leq |\bar{U}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}|b^u||b^v|dxdy$$

and by Young's inequality

$$\int_{-1}^{1}\int_{0}^{d}\bar{U}^{'}b^ub^vdxdy \leq \frac{1}{a_3}|\bar{U}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}(b^u)^2dxdy + a_3|\bar{U}^{'}|_{max}\int_{-1}^{1}\int_{0}^{d}(b^v)^2dxdy$$

and by equations (D-11) and (D-12)

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d}\bar{U}^{'}b^ub^vdxdy &\leq 2a_3|\bar{U}^{'}|_{max}(1+2b_3)\int_{0}^{d}(b_{top\_wall}^v)^2dx \\ &+ |\bar{U}^{'}|_{max}\left\{\frac{2}{a_3}\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy + 2a_3(1+\frac{2}{b_3})\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2dxdy\right\}\end{aligned} \tag{D-15}$$

Equation (47) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}\bar{b}^{'}\left(ub^v - vb^u\right)dxdy \leq \frac{N}{R_m}|\bar{b}^{'}|_{max}\left\{\int_{-1}^{1}\int_{0}^{d}|u||b^v|dxdy + \int_{-1}^{1}\int_{0}^{d}|v||b^u|dxdy\right\}$$

and by Young's inequality

$$\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}\bar{b}'\left(ub^v - vb^u\right)dxdy \;\; \leq \;\; \frac{N}{R_m}\left\{\frac{1}{a_4}|\bar{b}'|_{max}\int_{-1}^{1}\int_{0}^{d}u^2dxdy + a_4|\bar{b}'|_{max}\int_{-1}^{1}\int_{0}^{d}(b^v)^2dxdy\right\}$$
$$+ \;\; \frac{N}{R_m}\left\{\frac{1}{a_5}|\bar{b}'|_{max}\int_{-1}^{1}\int_{0}^{d}(b^u)^2dxdy + a_5|\bar{b}'|_{max}\int_{-1}^{1}\int_{0}^{d}v^2dxdy\right\}$$

and by equations (D-9)–(D-12)

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}\bar{b}'\left(ub^v - vb^u\right)dxdy \leq \frac{N}{R_m}&\left\{2a_4|\bar{b}'|_{max}(1+2b_4)\int_{0}^{d}(b^v_{top\_wall})^2dx\right.\\ &+|\bar{b}'|_{max}\left.\left\{\frac{2}{a_4}\int_{-1}^{1}\int_{0}^{d}u_y^2dxdy + 2a_4(1+\frac{2}{b_4})\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2dxdy\right\}\right\}\\ &+\frac{N}{R_m}\left\{2a_5|\bar{b}'|_{max}(1+2b_5)\int_{0}^{d}(v_{wall})^2dx\right.\\ &+|\bar{b}'|_{max}\left.\left\{\frac{2}{a_5}\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy + 2a_5(1+\frac{2}{b_5})\int_{-1}^{1}\int_{0}^{d}v_y^2dxdy\right\}\right\}\end{aligned} \quad \text{(D-16)}$$

Equation (48) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}\bar{b}\left(b_x^v - b_y^u\right)vdxdy \leq \frac{N}{R_m}|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}\left(|b_x^v| + |b_y^u|\right)|v|dxdy$$

and by Young's inequality

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}\bar{b}\left(b_x^v - b_y^u\right)vdxdy \;\; \leq \;\; \frac{N}{R_m}&\left\{\frac{1}{a_6}|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^2 + (b_y^u)^2\right]dxdy\right.\\ &\left.+\,2a_6|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}v^2dxdy\right\}\end{aligned}$$

and by equation (D-10)

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}\bar{b}\left(b_x^v - b_y^u\right)vdxdy \leq \frac{N}{R_m}&\left\{\frac{1}{a_6}|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^2 + (b_y^u)^2\right]dxdy\right.\\ &+2a_6|\bar{b}|_{max}\left.\left\{2(1+2b_6)\int_{0}^{d}v_{wall}^2dx + 2(1+\frac{2}{b_6})\int_{-1}^{1}\int_{0}^{d}v_y^2dxdy\right\}\right\}\end{aligned} \quad \text{(D-17)}$$

Equation (49) can be bounded as

$$-\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}B_o\left(b_x^v - b_y^u\right)udxdy \leq \frac{N}{R_m}B_o\int_{-1}^{1}\int_{0}^{d}\left(|b_x^v| + |b_y^u|\right)|u|dxdy$$

and by Young's inequality

$$\begin{aligned}-\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}B_o\left(b_x^v - b_y^u\right)udxdy \;\; \leq \;\; \frac{N}{R_m}&\left\{\frac{1}{a_7}B_o\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^2 + (b_y^u)^2\right]dxdy\right.\\ &\left.+\,2a_7B_o\int_{-1}^{1}\int_{0}^{d}u^2dxdy\right\}\end{aligned}$$

and by equation (D-9)

$$-\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}B_o\left(b_x^v-b_y^u\right)udxdy \leq \frac{N}{R_m}\left\{\frac{1}{a_7}B_o\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^2+(b_y^u)^2\right]dxdy + 2a_7B_o2\int_{-1}^{1}\int_{0}^{d}u_y^2dxdy\right\} \quad \text{(D-18)}$$

Equation (50) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d}\bar{b}\left(b^uu_x+b^vv_x\right)dxdy \leq |\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}\left(|b^u||u_x|+|b^v||v_x|\right)dxdy$$

and by Young's inequality

$$\int_{-1}^{1}\int_{0}^{d}\bar{b}\left(b^uu_x+b^vv_x\right)dxdy \leq \frac{1}{a_8}|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}u_x^2dxdy + a_8|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}(b^u)^2dxdy + \frac{1}{a_9}|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}v_x^2dxdy + a_9|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}(b^v)^2dxdy$$

and by equations (D-11) and (D-12)

$$\int_{-1}^{1}\int_{0}^{d}\bar{b}\left(b^uu_x+b^vv_x\right)dxdy \leq \frac{1}{a_8}|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}u_x^2dxdy + \frac{1}{a_9}|\bar{b}|_{max}\int_{-1}^{1}\int_{0}^{d}v_x^2dxdy + a_8|\bar{b}|_{max}2\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy + a_9|\bar{b}|_{max}\left\{2(1+2b_9)\int_{0}^{d}(b_{top\_wall}^v)^2dx + 2(1+\frac{2}{b_9})\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2dxdy\right\} \quad \text{(D-19)}$$

Equation (51) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d}B_o\left(b^uu_y+b^vv_y\right)dxdy \leq B_o\int_{-1}^{1}\int_{0}^{d}\left(|b^u||u_y|+|b^v||v_y|\right)dxdy$$

and by Young's inequality

$$\int_{-1}^{1}\int_{0}^{d}B_o\left(b^uu_y+b^vv_y\right)dxdy \leq \frac{1}{a_{10}}B_o\int_{-1}^{1}\int_{0}^{d}u_y^2dxdy + a_{10}B_o\int_{-1}^{1}\int_{0}^{d}(b^u)^2dxdy + \frac{1}{a_{11}}B_o\int_{-1}^{1}\int_{0}^{d}v_y^2dxdy + a_{11}B_o\int_{-1}^{1}\int_{0}^{d}(b^v)^2dxdy$$

and by equations (D-11) and (D-12)

$$\int_{-1}^{1}\int_{0}^{d}B_o\left(b^uu_y+b^vv_y\right)dxdy \leq \frac{1}{a_{10}}B_o\int_{-1}^{1}\int_{0}^{d}u_y^2dxdy + \frac{1}{a_{11}}B_o\int_{-1}^{1}\int_{0}^{d}v_y^2dxdy + a_{10}B_o2\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy + a_{11}B_o\left\{2(1+2b_{11})\int_{0}^{d}(b_{top\_wall}^v)^2dx + 2(1+\frac{2}{b_{11}})\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2dxdy\right\} \quad \text{(D-20)}$$

Equation (52) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}b^u\left(b_x^v-b_y^u\right)vdxdy \leq \frac{N}{R_m}\int_{-1}^{1}\int_{0}^{d}\left(|b_x^vb^u|+|b_y^ub^u|\right)|v|dxdy$$

and by Young's inequality

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}b^u\left(b_x^v-b_y^u\right)vdxdy &\leq \frac{N}{R_m}\left\{\frac{2}{a_{12}}\int_{-1}^{1}\int_{0}^{d}v^2dxdy\right. \\
&\left. + a_{12}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v b^u)^2+(b_y^u b^u)^2\right]dxdy\right\} \\
&\leq \frac{N}{R_m}\left\{\frac{2}{a_{12}}\int_{-1}^{1}\int_{0}^{d}v^2dxdy\right. \\
&\left. + a_{12}\left\{\frac{2}{c_{12}}\int_{-1}^{1}\int_{0}^{d}(b^u)^4dxdy + c_{12}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^4+(b_y^u)^4\right]dxdy\right\}\right\}
\end{aligned}$$

and by equation (D-10)

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}b^u\left(b_x^v-b_y^u\right)vdxdy &\leq \frac{N}{R_m}\left\{\frac{2}{a_{12}}\left\{2(1+2b_{12})\int_{0}^{d}v_{wall}^2dx+2(1+\frac{2}{b_{12}})\int_{-1}^{1}\int_{0}^{d}v_y^2dxdy\right\}\right. \\
&+a_{12}c_{12}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^4+(b_y^u)^4\right]dxdy \\
&\left.+a_{12}\frac{2}{c_{12}}\int_{-1}^{1}\int_{0}^{d}(b^u)^4dxdy\right\}
\end{aligned}$$

Considering that

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{d}(b^u)^4dxdy &\leq 2\int_{-1}^{1}\int_{0}^{d}(b^u)^2dxdy\int_{-1}^{1}\int_{0}^{d}\left[(b_x^u)^2+b_y^u)^2\right]dxdy \\
&\leq 2\left\{2\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy\right\}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^u)^2+b_y^u)^2\right]dxdy \\
&\leq 2\left\{\left(\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy\right)^2+\left(\int_{-1}^{1}\int_{0}^{d}\left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2\right\} \qquad \text{(D-21)}
\end{aligned}$$

we can finally write

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}b^u\left(b_x^v-b_y^u\right)vdxdy \leq \frac{N}{R_m}&\left\{\frac{2}{a_{12}}\left\{2(1+2b_{12})\int_{0}^{d}v_{wall}^2dx+2(1+\frac{2}{b_{12}})\int_{-1}^{1}\int_{0}^{d}v_y^2dxdy\right\}\right. \\
&+a_{12}c_{12}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^4+(b_y^u)^4\right]dxdy \\
&\left.+\frac{2a_{12}}{c_{12}}2\left\{\left(\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy\right)^2+\left(\int_{-1}^{1}\int_{0}^{d}\left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2\right\}\right\} \qquad \text{(D-22)}
\end{aligned}$$

Equation (53) can be bounded as

$$-\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}b^v\left(b_x^v-b_y^u\right)udxdy \leq \frac{N}{R_m}\int_{-1}^{1}\int_{0}^{d}\left(|b_x^v u|+|b_y^u u|\right)|b^v|dxdy$$

and by Young's inequality

$$-\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}b^v\left(b_x^v-b_y^u\right)udxdy \le \frac{N}{R_m}\left\{\frac{2}{a_{13}}\int_{-1}^{1}\int_{0}^{d}(b^v)^2dxdy + a_{13}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v u)^2+(b_y^u u)^2\right]dxdy\right\}$$

$$\le \frac{N}{R_m}\frac{2}{a_{13}}\int_{-1}^{1}\int_{0}^{d}(b^v)^2dxdy + \frac{N}{R_m}\,a_{13}\left\{\frac{2}{c_{13}}\int_{-1}^{1}\int_{0}^{d}u^4dxdy + c_{13}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^4+(b_y^u)^4\right]dxdy\right\}\Bigg\}$$

and by equation (D-12)

$$-\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}b^v\left(b_x^v-b_y^u\right)udxdy \le \frac{N}{R_m}\frac{2}{a_{13}}\left\{2(1+2b_{13})\int_{0}^{d}(b^v_{top\_wall})^2dx + 2(1+\frac{2}{b_{13}})\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2dxdy\right\} + \frac{N}{R_m}\,a_{13}\left\{\frac{2}{c_{13}}\int_{-1}^{1}\int_{0}^{d}u^4dxdy + c_{13}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^4+(b_y^u)^4\right]dxdy\right\}\Bigg\}$$

Considering that

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d}u^4dxdy &\le 2\int_{-1}^{1}\int_{0}^{d}u^2dxdy\int_{-1}^{1}\int_{0}^{d}\left[u_x^2+u_y^2\right]dxdy\\ &\le 2\left\{2\int_{-1}^{1}\int_{0}^{d}u_y^2dx\right\}\int_{-1}^{1}\int_{0}^{d}\left[u_x^2+u_y^2\right]dxdy\\ &\le 2\left\{\left(\int_{-1}^{1}\int_{0}^{d}u_y^2dxdy\right)^2+\left(\int_{-1}^{1}\int_{0}^{d}\left[u_x^2+u_y^2\right]dxdy\right)^2\right\}\end{aligned} \quad \text{(D-23)}$$

and using equation (D-12) we can finally write

$$\begin{aligned}-\int_{-1}^{1}\int_{0}^{d}\frac{N}{R_m}b^v\left(b_x^v-b_y^u\right)udxdy \le \frac{N}{R_m}&\left\{\frac{2}{a_{13}}\left\{2(1+2b_{13})\int_{0}^{d}(b^v_{top\_wall})^2dx+2(1+\frac{2}{b_{13}})\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2dxdy\right\}\right.\\ &+\frac{2a_{13}}{c_{13}}2\left\{\left(\int_{-1}^{1}\int_{0}^{d}u_y^2dxdy\right)^2+\left(\int_{-1}^{1}\int_{0}^{d}\left[u_x^2+u_y^2\right]dxdy\right)^2\right\}\\ &\left.+a_{13}c_{13}\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^4+(b_y^u)^4\right]dxdy\right\}\end{aligned} \quad \text{(D-24)}$$

Equation (54) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d}b^ub^uu_xdxdy \le \int_{-1}^{1}\int_{0}^{d}|u_x|(b^u)^2dxdy$$

and by Young's inequality

$$\int_{-1}^{1}\int_{0}^{d}b^ub^uu_xdxdy \le \frac{1}{a_{14}}\int_{-1}^{1}\int_{0}^{d}u_x^2dxdy + a_{14}\int_{-1}^{1}\int_{0}^{d}(b^u)^4dxdy$$

and using equation (D-21) we can finally write

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d}b^ub^uu_xdxdy \le &\frac{1}{a_{14}}\int_{-1}^{1}\int_{0}^{d}u_x^2dxdy\\ &+2a_{14}\left\{\left(\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy\right)^2+\left(\int_{-1}^{1}\int_{0}^{d}\left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2\right\}\end{aligned} \quad \text{(D-25)}$$

Equation (55) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d} b^u b^v v_x dxdy \le \int_{-1}^{1}\int_{0}^{d} |v_x||b^u b^v| dxdy$$

and by Young's inequality

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d} b^u b^v v_x dxdy &\le \frac{1}{a_{15}}\int_{-1}^{1}\int_{0}^{d} v_x^2 dxdy + a_{15}\int_{-1}^{1}\int_{0}^{d} (b^u b^v)^2 dxdy \\ &\le \frac{1}{a_{15}}\int_{-1}^{1}\int_{0}^{d} v_x^2 dxdy \\ &\quad + a_{15}\left\{\frac{1}{c_{15}}\int_{-1}^{1}\int_{0}^{d} (b^u)^4 dxdy + c_{15}\int_{-1}^{1}\int_{0}^{d} (b^v)^4 dxdy\right\}\end{aligned}$$

Using equations (D-21) and considering that

$$\begin{aligned}&\int_{-1}^{1}\int_{0}^{d} (b^v)^4 dxdy \le 2\int_{-1}^{1}\int_{0}^{d} (b^v)^2 dxdy \int_{-1}^{1}\int_{0}^{d} \left[(b_x^v)^2 + (b_y^v)^2\right] dxdy \\ &\le 2\left\{2\left(1+2b\right)\int_{0}^{d} (b_{top\_wall}^v)^2 dx + 2\left(1+\frac{2}{b}\right)\int_{-1}^{1}\int_{0}^{d} (b_y^v)^2 dxdy\right\}\int_{-1}^{1}\int_{0}^{d} \left[(b_x^v)^2 + (b_y^v)^2\right] dxdy \\ &\le 2\left(1+2b\right)\left\{\left(\int_{0}^{d} (b_{top\_wall}^v)^2 dx\right)^2 + \left(\int_{-1}^{1}\int_{0}^{d} \left[(b_x^v)^2 + (b_y^v)^2\right] dxdy\right)^2\right\} \\ &\quad + 2\left(1+\frac{2}{b}\right)\left\{\left(\int_{-1}^{1}\int_{0}^{d} (b_y^v)^2 dxdy\right)^2 + \left(\int_{-1}^{1}\int_{0}^{d} \left[(b_x^v)^2 + (b_y^v)^2\right] dxdy\right)^2\right\}\end{aligned} \quad \text{(D-26)}$$

we can finally write

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d} b^u b^v v_x dxdy &\le \frac{1}{a_{15}}\int_{-1}^{1}\int_{0}^{d} v_x^2 dxdy \\ &+ \frac{2a_{15}}{c_{15}}\left\{\left(\int_{-1}^{1}\int_{0}^{d} (b_y^u)^2 dxdy\right)^2 + \left(\int_{-1}^{1}\int_{0}^{d} \left[(b_x^u)^2 + b_y^u)^2\right] dxdy\right)^2\right\} \\ &+ 2a_{15}c_{15}\left\{\left(1+\frac{2}{b_{15}}\right)\left(\int_{-1}^{1}\int_{0}^{d} (b_y^v)^2 dxdy\right)^2 + (1+2b_{15})\left(\int_{-1}^{1}\int_{0}^{d} (b_{top\_wall}^v)^2 dxdy\right)^2\right. \\ &\left. + \left(2+2b_{15}+\frac{2}{b_{15}}\right)\left(\int_{-1}^{1}\int_{0}^{d} \left[(b_x^v)^2 + (b_y^v)^2\right] dxdy\right)^2\right\}\end{aligned} \quad \text{(D-27)}$$

Equation (56) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d} b^v b^v v_y dxdy \le \int_{-1}^{1}\int_{0}^{d} |v_y|(b^v)^2 dxdy$$

and by Young's inequality

$$\int_{-1}^{1}\int_{0}^{d} b^v b^v v_y dxdy \le \frac{1}{a_{16}}\int_{-1}^{1}\int_{0}^{d} v_y^2 dxdy + a_{16}\int_{-1}^{1}\int_{0}^{d} (b^v)^4 dxdy$$

and using equation (D-26) we can finally write

$$\int_{-1}^{1}\int_{0}^{d} b^v b^v v_y dxdy \leq \frac{1}{a_{16}}\int_{-1}^{1}\int_{0}^{d} v_y^2 dxdy$$
$$+2a_{16}\left\{\left(1+\frac{2}{b_{16}}\right)\left(\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2 dxdy\right)^2 + (1+2b_{16})\left(\int_{-1}^{1}\int_{0}^{d}(b_{top\_wall}^v)^2 dxdy\right)^2\right.$$
$$\left.+\left(2+2b_{16}+\frac{2}{b_{16}}\right)\left(\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^2+(b_y^v)^2\right]dxdy\right)^2\right\} \qquad \text{(D-28)}$$

Equation (57) can be bounded as

$$\int_{-1}^{1}\int_{0}^{d} b^u b^v u_y dxdy \leq \int_{-1}^{1}\int_{0}^{d} |u_y||b^u b^v| dxdy$$

and by Young's inequality

$$\begin{aligned}\int_{-1}^{1}\int_{0}^{d} b^u b^v u_y dxdy \quad &\leq \quad \frac{1}{a_{17}}\int_{-1}^{1}\int_{0}^{d} u_y^2 dxdy + a_{17}\int_{-1}^{1}\int_{0}^{d}(b^u b^v)^2 dxdy \\ &\leq \quad \frac{1}{a_{17}}\int_{-1}^{1}\int_{0}^{d} u_y^2 dxdy \\ &\quad\quad +a_{17}\left\{\frac{1}{c_{17}}\int_{-1}^{1}\int_{0}^{d}(b^u)^4 dxdy + c_{15}\int_{-1}^{1}\int_{0}^{d}(b^v)^4 dxdy\right\}\end{aligned}$$

and using equations (D-21) and (D-26) we can finally write

$$\int_{-1}^{1}\int_{0}^{d} b^u b^v u_y dxdy \leq \frac{1}{a_{17}}\int_{-1}^{1}\int_{0}^{d} u_y^2 dxdy$$
$$+\frac{2a_{17}}{c_{17}}\left\{\left(\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2 dxdy\right)^2 + \left(\int_{-1}^{1}\int_{0}^{d}\left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2\right\}$$
$$+2a_{17}c_{17}\left\{\left(1+\frac{2}{b_{17}}\right)\left(\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2 dxdy\right)^2 + (1+2b_{17})\left(\int_{-1}^{1}\int_{0}^{d}(b_{top\_wall}^v)^2 dxdy\right)^2\right.$$
$$\left.+\left(2+2b_{17}+\frac{2}{b_{17}}\right)\left(\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^2+(b_y^v)^2\right]dxdy\right)^2\right\} \qquad \text{(D-29)}$$

Taking into account equations (D-13), (D-14), (D-15), (D-16), (D-17), (D-18), (D-19), (D-20), (D-22), (D-24), (D-25), (D-27), (D-28) and (D-29), we can write

$$
\begin{aligned}
|g(\mathbf{v},\mathbf{B})| \quad \leq \quad & k_1 \left\{ 2a_1|\bar{U}^{'}|_{max}(1+2b_1)\int_0^d v_{wall}^2 dx \right. \\
& \left. + |\bar{U}^{'}|_{max}\left\{ \frac{2}{a_1}\int_{-1}^{1}\int_0^d u_y^2 dxdy + 2a_1(1+\frac{2}{b_1})\int_{-1}^{1}\int_0^d v_y^2 dxdy \right\}\right\} \qquad \text{(D-30)} \\
& + k_2 \left\{ 2a_2|\bar{b}^{'}|_{max}(1+2b_2)\int_0^d v_{wall}^2 dx \right. \\
& \left. + |\bar{b}^{'}|_{max}\left\{ \frac{2}{a_2}\int_{-1}^{1}\int_0^d (b_y^u)^2 dxdy + 2a_2(1+\frac{2}{b_2})\int_{-1}^{1}\int_0^d v_y^2 dxdy \right\}\right\} \qquad \text{(D-31)} \\
& + k_2 \left\{ 2a_3|\bar{U}^{'}|_{max}(1+2b_3)\int_0^d (b_{top\_wall}^v)^2 dx \right. \\
& \left. + |\bar{U}^{'}|_{max}\left\{ \frac{2}{a_3}\int_{-1}^{1}\int_0^d (b_y^u)^2 dxdy + 2a_3(1+\frac{2}{b_3})\int_{-1}^{1}\int_0^d (b_y^v)^2 dxdy \right\}\right\} \qquad \text{(D-32)} \\
& + k_1 \left\{ \frac{N}{R_m}\left\{ 2a_4|\bar{b}^{'}|_{max}(1+2b_4)\int_0^d (b_{top\_wall}^v)^2 dx \right.\right. \\
& \left. + |\bar{b}^{'}|_{max}\left\{ \frac{2}{a_4}\int_{-1}^{1}\int_0^d u_y^2 dxdy + 2a_4(1+\frac{2}{b_4})\int_{-1}^{1}\int_0^d (b_y^v)^2 dxdy \right\}\right\} \\
& + \frac{N}{R_m}\left\{ 2a_5|\bar{b}^{'}|_{max}(1+2b_5)\int_0^d v_{wall}^2 dx \right. \\
& \left.\left. + \ |\bar{b}^{'}|_{max}\left\{ \frac{2}{a_5}\int_{-1}^{1}\int_0^d (b_y^u)^2 dxdy + 2a_5(1+\frac{2}{b_5})\int_{-1}^{1}\int_0^d v_y^2 dxdy \right\}\right\}\right\} \qquad \text{(D-33)} \\
& + k_1 \left\{ \frac{N}{R_m}\left\{ \frac{1}{a_6}|\bar{b}|_{max}\int_{-1}^{1}\int_0^d \left[(b_x^v)^2 + (b_y^u)^2\right] dxdy \right.\right. \\
& \left.\left. + \ 2a_6|\bar{b}|_{max}\left\{ 2(1+2b_6)\int_0^d v_{wall}^2 dxdy + 2(1+\frac{2}{b_6})\int_{-1}^{1}\int_0^d v_y^2 dxdy \right\}\right\}\right\} \qquad \text{(D-34)} \\
& + k_1 \left\{ \frac{N}{R_m}\left\{ \frac{1}{a_7}B_o\int_{-1}^{1}\int_0^d \left[(b_x^v)^2 + (b_y^u)^2\right] dxdy \right.\right. \\
& \left.\left. + \ 2a_7 B_o 2\int_{-1}^{1}\int_0^d u_y^2 dxdy \right\}\right\} \qquad \text{(D-35)} \\
& + k_2 \left\{ \frac{1}{a_8}|\bar{b}|_{max}\int_{-1}^{1}\int_0^d u_x^2 dxdy + \frac{1}{a_9}|\bar{b}|_{max}\int_{-1}^{1}\int_0^d v_x^2 dxdy \right. \\
& + a_8|\bar{b}|_{max}2\int_{-1}^{1}\int_0^d (b_y^u)^2 dxdy \\
& \left. + \ a_9|\bar{b}|_{max}\left\{ 2(1+2b_9)\int_0^d (b_{top\_wall}^v)^2 dx + 2(1+\frac{2}{b_9})\int_{-1}^{1}\int_0^d (b_y^v)^2 dxdy \right\}\right\} \qquad \text{(D-36)}
\end{aligned}
$$

$$+k_2\left\{\frac{1}{a_{10}}B_o\int_{-1}^{1}\int_0^d u_y^2dxdy+\frac{1}{a_{11}}B_o\int_{-1}^{1}\int_0^d v_y^2dxdy\right.$$

$$+a_{10}B_o2\int_{-1}^{1}\int_0^d (b_y^u)^2dxdy$$

$$\left.+\ a_{11}B_o\left\{2(1+2b_{11})\int_0^d (b^v_{top\_wall})^2dx+2(1+\frac{2}{b_{11}})\int_{-1}^{1}\int_0^d (b_y^v)^2dxdy\right\}\right\} \tag{D-37}$$

$$+k_1\left\{\frac{N}{R_m}\left\{\frac{2}{a_{12}}\left\{2(1+2b_{12})\int_0^d v_{wall}^2dxdy+2(1+\frac{2}{b_{12}})\int_{-1}^{1}\int_0^d v_y^2dxdy\right\}\right.\right.$$

$$+a_{12}c_{12}\int_{-1}^{1}\int_0^d \left[(b_x^v)^4+(b_y^u)^4\right]dxdy$$

$$\left.\left.+\frac{2a_{12}}{c_{12}}2\left\{\left(\int_{-1}^{1}\int_0^d (b_y^u)^2dxdy\right)^2+\left(\int_{-1}^{1}\int_0^d \left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2\right\}\right\}\right\} \tag{D-38}$$

$$+k_1\left\{\frac{N}{R_m}\left\{\frac{2}{a_{13}}\left\{2(1+2b_{13})\int_0^d (b^v_{top\_wall})^2dx+2(1+\frac{2}{b_{13}})\int_{-1}^{1}\int_0^d (b_y^v)^2dxdy\right\}\right.\right.$$

$$+\frac{2a_{13}}{c_{13}}2\left\{\left(\int_{-1}^{1}\int_0^d u_y^2dxdy\right)^2+\left(\int_{-1}^{1}\int_0^d \left[u_x^2+u_y^2\right]dxdy\right)^2\right\}$$

$$\left.\left.+\ a_{13}c_{13}\int_{-1}^{1}\int_0^d \left[(b_x^v)^4+(b_y^u)^4\right]dxdy\right\}\right\} \tag{D-39}$$

$$+k_2\left\{\frac{1}{a_{14}}\int_{-1}^{1}\int_0^d u_x^2dxdy\right.$$

$$\left.+2a_{14}\left\{\left(\int_{-1}^{1}\int_0^d (b_y^u)^2dxdy\right)^2+\left(\int_{-1}^{1}\int_0^d \left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2\right\}\right\} \tag{D-40}$$

$$+k_2\left\{\frac{1}{a_{15}}\int_{-1}^{1}\int_0^d v_x^2dxdy\right.$$

$$+\frac{2a_{15}}{c_{15}}\left\{\left(\int_{-1}^{1}\int_0^d (b_y^u)^2dxdy\right)^2+\left(\int_{-1}^{1}\int_0^d \left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2\right\}$$

$$+2a_{15}c_{15}\left\{\left(1+\frac{2}{b_{15}}\right)\left(\int_{-1}^{1}\int_0^d (b_y^v)^2dxdy\right)^2+(1+2b_{15})\left(\int_{-1}^{1}\int_0^d (b^v_{top\_wall})^2dxdy\right)^2\right.$$

$$\left.\left.+\left(2+2b_{15}+\frac{2}{b_{15}}\right)\left(\int_{-1}^{1}\int_0^d \left[(b_x^v)^2+(b_y^v)^2\right]dxdy\right)^2\right\}\right\} \tag{D-41}$$

$$+k_2\left\{\frac{1}{a_{16}}\int_{-1}^{1}\int_0^d v_y^2dxdy\right.$$

$$+2a_{16}\left\{\left(1+\frac{2}{b_{16}}\right)\left(\int_{-1}^{1}\int_0^d (b_y^v)^2dxdy\right)^2+(1+2b_{16})\left(\int_{-1}^{1}\int_0^d (b^v_{top\_wall})^2dxdy\right)^2\right.$$

$$\left.\left.+\left(2+2b_{16}+\frac{2}{b_{16}}\right)\left(\int_{-1}^{1}\int_0^d \left[(b_x^v)^2+(b_y^v)^2\right]dxdy\right)^2\right\}\right\} \tag{D-42}$$

$$+k_2\left\{\frac{1}{a_{17}}\int_{-1}^{1}\int_{0}^{d}u_y^2dxdy\right.$$

$$+\frac{2a_{17}}{c_{17}}\left\{\left(\int_{-1}^{1}\int_{0}^{d}(b_y^u)^2dxdy\right)^2+\left(\int_{-1}^{1}\int_{0}^{d}\left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2\right\}$$

$$+2a_{17}c_{17}\left\{\left(1+\frac{2}{b_{17}}\right)\left(\int_{-1}^{1}\int_{0}^{d}(b_y^v)^2dxdy\right)^2+(1+2b_{17})\left(\int_{-1}^{1}\int_{0}^{d}(b_{top\_wall}^v)^2dxdy\right)^2\right.$$

$$\left.\left.+\left(2+2b_{17}+\frac{2}{b_{17}}\right)\left(\int_{-1}^{1}\int_{0}^{d}\left[(b_x^v)^2+(b_y^v)^2\right]dxdy\right)^2\right\}\right\} \quad \text{(D-43)}$$

Defining

$$h_1 = \left(\frac{k_2}{a_8}|\bar{b}|_{max}+\frac{k_2}{a_{14}}\right) \quad \text{(D-44)}$$

$$h_2 = \left(k_1|\bar{U}'|_{max}\frac{2}{a_1}+k_1\frac{N}{R_m}|\bar{b}'|_{max}\frac{2}{a_4}+k_1\frac{N}{R_m}2a_7B_o2+\frac{k_2}{a_{10}}B_o+\frac{k_2}{a_{17}}\right) \quad \text{(D-45)}$$

$$h_3 = \left(k_2\frac{1}{a_9}|\bar{b}|_{max}+\frac{k_2}{a_{15}}\right) \quad \text{(D-46)}$$

$$h_4 = \left(k_1|\bar{U}'|_{max}2a_1(1+\frac{2}{b_1})+k_2|\bar{b}'|_{max}2a_2(1+\frac{2}{b_2})+k_1\frac{N}{R_m}|\bar{b}'|_{max}2a_5(1+\frac{2}{b_5})\right.$$
$$\left.+k_1\frac{N}{R_m}2a_6|\bar{b}|_{max}2(1+\frac{2}{b_6})+\frac{k_2}{a_{11}}B_o+k_1\frac{N}{R_m}2(1+\frac{2}{b_{12}})+\frac{k_2}{a_{16}}\right) \quad \text{(D-47)}$$

$$h_5 = \left(k_1\frac{N}{R_m}\frac{1}{a_6}|\bar{b}|_{max}+k_1\frac{N}{R_m}\frac{1}{a_7}B_o\right) \quad \text{(D-48)}$$

$$h_6 = \left(k_2|\bar{b}'|_{max}\frac{2}{a_2}+k_2|\bar{U}'|_{max}\frac{2}{a_3}+k_1\frac{N}{R_m}|\bar{b}'|_{max}\frac{2}{a_5}+k_1\frac{N}{R_m}\frac{1}{a_6}|\bar{b}|_{max}\right.$$
$$\left.+k_1\frac{N}{R_m}\frac{1}{a_7}B_o+k_2a_8|\bar{b}|_{max}2+k_2a_{10}B_o2\right) \quad \text{(D-49)}$$

$$h_7 = \left(k_2|\bar{U}'|_{max}2a_3(1+\frac{2}{b_3})+k_1\frac{N}{R_m}|\bar{b}'|_{max}2a_4(1+\frac{2}{b_4})+k_2a_9|\bar{b}|_{max}2(1+\frac{2}{b_9})\right.$$
$$\left.+k_2a_{11}B_o2(1+\frac{2}{b_{11}})+k_1\frac{N}{R_m}2(1+\frac{2}{b_{13}})\right) \quad \text{(D-50)}$$

$$h_8 = k_1\frac{N}{R_m}\frac{2a_{13}}{c_{13}}2 \quad \text{(D-51)}$$

$$h_9 = \left(k_1\frac{N}{R_m}\frac{2a_{12}}{c_{12}}2+k_22a_{14}+k_2\frac{2a_{15}}{c_{15}}+k_2\frac{2a_{17}}{c_{17}}\right) \quad \text{(D-52)}$$

$$h_{10} = \left(k_22a_{15}c_{15}\left(1+\frac{2}{b_{15}}\right)+k_22a_{16}\left(1+\frac{2}{b_{16}}\right)+k_22a_{17}c_{17}\left(1+\frac{2}{b_{17}}\right)\right) \quad \text{(D-53)}$$

$$h_{11} = \left(k_22a_{15}c_{15}\left(2+2b_{15}+\frac{2}{b_{15}}\right)+k_22a_{16}\left(2+2b_{16}+\frac{2}{b_{16}}\right)\right.$$
$$\left.+k_22a_{17}c_{17}\left(2+2b_{17}+\frac{2}{b_{17}}\right)\right) \quad \text{(D-54)}$$

$$h_{12} = \left(k_1\frac{N}{R_m}\frac{2a_{12}}{c_{12}}2 + k_2 2a_{14} + k_2\frac{2a_{15}}{c_{15}} + k_2\frac{2a_{17}}{c_{17}}\right) \quad \text{(D-55)}$$

$$h_{13} = k_1\frac{N}{R_m}a_{12}c_{12} + k_1\frac{N}{R_m}a_{13}c_{13} \quad \text{(D-56)}$$

$$\begin{aligned} h_{14} = & \left(k_1 2a_1|\bar{U}'|_{max}(1+2b_1) + k_2 2a_2|\bar{b}'|_{max}(1+2b_2) + k_1\frac{N}{R_m}2a_5|\bar{b}'|_{max}(1+2b_5)\right. \\ & \left. +k_1\frac{N}{R_m}2a_6|\bar{b}|_{max}2(1+2b_6) + k_1\frac{N}{R_m}\frac{2}{a_{12}}2(1+2b_{12})\right) \end{aligned} \quad \text{(D-57)}$$

$$\begin{aligned} h_{15} = & \left(k_2 2a_3|\bar{U}'|_{max}(1+2b_3) + k_1\frac{N}{R_m}2a_4|\bar{b}'|_{max}(1+2b_4) + k_2 a_9|\bar{b}|_{max}2(1+2b_9)\right. \\ & \left. +k_2 a_{11}B_o 2(1+2b_{11}) + k_1\frac{N}{R_m}\frac{2}{a_{13}}2(1+2b_{13})\right) \end{aligned} \quad \text{(D-58)}$$

$$h_{16} = (k_2 2a_{15}c_{15}(1+2b_{15}) + k_2 2a_{16}(1+2b_{16}) + k_2 2a_{17}c_{17}(1+2b_{17})) \quad \text{(D-59)}$$

we write

$$\begin{aligned} |g(\mathbf{v},\mathbf{B})| \leq\; & h_1\int_{-1}^{1}\int_0^d u_x^2dxdy + h_2\int_{-1}^{1}\int_0^d u_y^2dxdy + h_3\int_{-1}^{1}\int_0^d v_x^2dxdy + h_4\int_{-1}^{1}\int_0^d v_y^2dxdy \\ & +h_5\int_{-1}^{1}\int_0^d (b_x^v)^2dxdy + h_6\int_{-1}^{1}\int_0^d (b_y^u)^2dxdy + h_7\int_{-1}^{1}\int_0^d (b_y^v)^2dxdy \\ & +h_8\left(\int_{-1}^{1}\int_0^d u_y^2dxdy\right)^2 + h_8\left(\int_{-1}^{1}\int_0^d \left[u_x^2+u_y^2\right]dxdy\right)^2 + h_9\left(\int_{-1}^{1}\int_0^d (b_y^u)^2dxdy\right)^2 \\ & +h_{10}\left(\int_{-1}^{1}\int_0^d (b_y^v)^2dxdy\right)^2 + h_{11}\left(\int_{-1}^{1}\int_0^d \left[(b_x^v)^2+(b_y^v)^2\right]dxdy\right)^2 \\ & +h_{12}\left(\int_{-1}^{1}\int_0^d \left[(b_x^u)^2+b_y^u)^2\right]dxdy\right)^2 + h_{13}\int_{-1}^{1}\int_0^d \left[(b_x^v)^4+(b_y^u)^4\right]dxdy \\ & +h_{14}\int_0^d v_{wall}^2dx + h_{15}\int_0^d (b_{top\_wall}^v)^2dx + h_{16}\left(\int_0^d (b_{top\_wall}^v)^2dx\right)^2 \end{aligned}$$

Defining $g_1 = \max(h_1,h_2,h_3,h_4,h_5,h_6,h_7)$, $g_2 = 6\max(h_8,h_9,h_{10},h_{11},h_{12},h_{13})$, $g_3 = h_{14}$, $g_4 = h_{15}$, $g_5 = h_{16}$, we can finally write

$$|g(\mathbf{v},\mathbf{B})| \leq g_1 m(\mathbf{v},\mathbf{B}) + g_2 m^2(\mathbf{v},\mathbf{B}) + g_3\int_0^d v_{wall}^2dx + g_4\int_0^d (b_{top\_wall}^v)^2dx + g_5\left(\int_0^d (b_{top\_wall}^v)^2dx\right)^2.$$

□

References

[1] Spong, E., Reizes, J., and Leonardi, E., "Efficiency improvements of electromagnetic flow control," *Heat and Fluid Flow*, Vol. 26, 2005, pp. 635–655.

[2] Choi, H., Moin, P., and Kim, J., "Active turbulence control for drag reduction in wall-bounded flows," *J. Fluid. Mech.*, Vol. 262, 1994, pp. 75.

[3] Berger, T., Kim, J., Lee, C., and Lim, J., "Turbulent boundary layer control utilizing the Lorentz force," *Phys. Fluids*, Vol. 12, 2000, pp. 631.

[4] Baker, J., Armaou, A., and Christofides, P., "Drag reduction in transitional linearized channel flow using distributed control," *Int. J. Control*, Vol. 75, No. 15, 2002, pp. 1213–1218.

[5] Singh, S. and Bandyopadhyay, P., "Linear Feedback control of boundary layer using electromagnetic microtiles," *Transactions of ASME*, Vol. 119, 1997, pp. 852–858.

[6] Poggie, J. and Gaitonde, D., "Magnetic control of flow past a blunt body: Numerical validation and exploration," *Physics of Fluids*, Vol. 14, No. 5, 2002, pp. 1720.

[7] Poggie, J. and Gaitonde, D., "Computational Studies of Magnetic Control in Hypersonic Flow," *Proceedings of the 39th Aerospace Sciences Meeting and Exhibit*, Reno, NV., 2001, AIAA 2001-0196.

[8] Dietiker, J.-F. and Hoffmann, K., "Boundary Layer Control in Magnetohydrodynamic Flows," *Proceedings of the 40th Aerospace Sciences Meeting and Exhibit*, Reno, NV., 2002, AIAA 2002-0130.

[9]Poggie, J. and Gaitonde, D., "Electrode Boundary Conditions in Magnetogasdynamic Flow Control," *Proceedings of the 40th Aerospace Sciences Meeting and Exhibit*, Reno, NV., 2002, AIAA 2002-0199.

[10]Gaitonde, D. and Poggie, J., "Elements of a Numerical Procedure for 3-D MGD Flow Control Analysis," *Proceedings of the 40th Aerospace Sciences Meeting and Exhibit*, Reno, NV., 2002, AIAA 2002-0198.

[11]Morley, N. B., Smolentsev, S., Barleon, L., Kirillov, I. R., and Takashi, M., "Liquid magnetohydrodynamics - recent progress and future directions for fusion," *Fusion Engineering and Design*, Vol. 51–52, 2000, pp. 701–713.

[12]http://www.techtransfer.anl.gov/techtour/nanofluids-summary.html.

[13]Kirillov, I., Reed, C., Barleon, L., and Miyazaki, K., "Present understanding of MHD and heat transfer phenomena for liquid metal blankets," *Fusion Engineering and Design*, Vol. 27, 1995, pp. 553–569.

[14]Balogh, A., Liu, W.-J., and Krstic, M., "Stability enhancement by boundary control in 2D channel flow," *IEEE Transactions on Automatic Control*, Vol. 46, No. 11, 2001, pp. 1696–1711.

[15]Aamo, O. M., Krstić, M., and Bewley, T. R., "Control of mixing by boundary feedback in 2D channel flow," *Automatica*, Vol. 39, 2003, pp. 1597–1606.

[16]Aamo, O. M., Balogh, A., and Krstic, M., "Optimal mixing by feedback in pipe flow," *Proceedings of the 15th IFAC World Congress on Automatic Control*, Barcelona, Spain, 2002.

[17]Balogh, A., Aamo, O. M., and Krstic, M., "Optimal mixing enhancement in 3D pipe flow," *IEEE Transactions on Control Systems Technology*, Vol. 13, No. 1, 2005, pp. 27–41.

[18]Schuster, E. and Krstić, M., "Inverse optimal boundary control of mixing in magnetohydrodynamic channel flows," *Proceedings of the 2003 IEEE Conference on Decision and Control*, Maui, Hawaii, December 2003, pp. 3966–71.

[19]Hartmann, J., "Theory of the laminar flow of an electrically conductive liquid in a homogeneous magnetic field," *Det Kgl. Danske Vidensk-abernes Selskab. Mathematisk-fysiske Meddelelser.*, Vol. XV(6), pp. 1–27.

[20]Müller, U. and Bühler, L., *Magnetofluiddynamics in Channels and Containers*, Springer, 2001.

[21]Powell, K. G., Roe, P. L., Myong, R. S., Gombosi, T., and Zeeuw, D. D., "An Upwind Scheme for Magnetohydrodynamics," June 1995, AIAA 95-1704-C.

[22]Damevin, H.-M. and Hoffmann, K. A., "Development of a Modified Runge-Kutta Scheme with TVD Limiters for Ideal Three-Dimensional Magnetogasdynamics," *Proceedings of the 32nd AIAA Plasmadynamics and Lasers Conference*, Anaheim, CA, June 2001, AIAA 2001-2739.

[23]Gaitonde, D. V., "Development of a Solver for 3-D Non-Ideal Magnetogasdynamics," *Proceedings of the 30th Plasmadynamics and Lasers Conference*, Norfolk, VA, June 1999, AIAA 99-3610.

[24]Cannup, P. W., "Resolution of Magnetogasdynamic Phenomena Usinga a Flux-Vector Splitting Method," *Fluids 2000*, Denver, CO, June 2000, AIAA 2000-2477.

[25]Toth, G. and Odstrcil, D., "omparison of Some Flux Corrected Transport and Total Variation Diminishing Numerical Schemes for Hydrodynamic and Magnetohydrodynamic Problems," *Journal of Computational Physics*, Vol. 128, No. 197, 1996, pp. pp. 82–100.

[26]Murawski, K. and Tanaka, T., "Modern Numerical Schemes for solving Magnetohydrodynamic Equations," *Astrophysics and Space Science*, Vol. 254, 1997, pp. 187–210.

[27]Balsara, D. S. and Spicer, D. S., "A staggered mesh algorithm using high order Godunov fluxes to ensure solenoidal magnetic fields in magnetohydrodynamic simulations," *Journal of Computational Physics*, Vol. 149, 1999, pp. 270–292.

[28]Dai, W. L. and Woodward, P. R., "A simple finite difference scheme for multidimensional magnetohydrodynamical equations," *Journal of Computational Physics*, Vol. 142, 1998, pp. 331–369.

[29]DeVore, C. R., "Flux-corrected transport techniques for multidimensional compressible magnetohydrodynamics," *Journal of Computational Physics*, Vol. 92, 1991, pp. 142–160.

[30]Evans, C. H. and Hawley, J. H., "Simulation of magnetohydrodynamic flows: a constrained transport method," *Astrophysics Journal*, Vol. 332, 1988, pp. 659–677.

[31]Canuto, C., *Spectral methods in fluid dynamics*, Springer Verlag, 1998.

[32]Morinishi, Y., Lund, T., Vasilyev, O., and Moin, P., "Fully conservative higher order finite difference schemes for incompressible flow," *Journal of Computational Physics*, Vol. 143, No. 1, 1998, pp. 90–124.

[33]Dukowicz, J. and Dvinsky, A., "Approximate factorization as a high order splitting for the implicit incompressible flow equations," *Journal of Computational Physics*, Vol. 102, No. 2, 1992, pp. 336–347.

[34]Bewley, T., *Optimal and robust control and estimation of transition, convection, and turbulence*, Stanford University thesis, 1999.

[35]Panton, R. L., *Incompressible flow*, Wiley, New York, 2nd ed., 1996.