



Brief paper

Sequential linear quadratic control of bilinear parabolic PDEs based on POD model reduction[☆]

Chao Xu^{a,b,*}, Yongsheng Ou^{a,c}, Eugenio Schuster^a

^a Department of Mechanical Engineering & Mechanics, Lehigh University, Bethlehem, PA 18015-1835, USA

^b Department of Control Science & Engineering, Zhejiang University, Hangzhou, Zhejiang 310027, China

^c Shenzhen Institutes of Advanced Technology, Chinese Academy of Sciences, Shenzhen, Guangdong 518055, China

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ABSTRACT

We present a framework to solve a finite-time optimal control problem for parabolic partial differential equations (PDEs) with diffusivity-interior actuators, which is motivated by the control of the current density profile in tokamak plasmas. The proposed approach is based on reduced order modeling (ROM) and successive optimal control computation. First we either simulate the parabolic PDE system or carry out experiments to generate data ensembles, from which we then extract the most energetic modes to obtain a reduced order model based on the proper orthogonal decomposition (POD) method and Galerkin projection. The obtained reduced order model corresponds to a bilinear control system. Based on quasi-linearization of the optimality conditions derived from Pontryagin's maximum principle, and stated as a two boundary value problem, we propose an iterative scheme for suboptimal closed-loop control. We take advantage of linear synthesis methods in each iteration step to construct a sequence of controllers. The convergence of the controller sequence is proved in appropriate functional spaces. When compared with previous iterative schemes for optimal control of bilinear systems, the proposed scheme avoids repeated numerical computation of the Riccati equation and therefore reduces significantly the number of ODEs that must be solved at each iteration step. A numerical simulation study shows the effectiveness of this approach.

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1. Introduction

Many dissipative physical systems can be modeled by parabolic equations and the research concerning the associated control problems has a long history in the field of distributed parameter system theory (e.g., Bensoussan et al., 2006). Physical actuation can appear in parabolic partial differential equations (PDEs) in three different ways: source terms (interior control), boundary conditions (boundary control) and diffusivity coefficients (diffusivity control). Topics concerning interior and boundary control have been studied extensively, and many approaches to PDE control have been

proposed. For instance, using the inertial manifold theory, most dissipative PDE systems with interior actuators can be approximated asymptotically by appropriate finite dimensional ordinary differential equation (ODEs) systems. These ODE systems are used as the basis for the synthesis of controllers based on finite dimensional control theory (e.g., Christofides, 2001 and references therein). Similarly, backstepping has been proved as a powerful approach to the control of PDE systems with boundary actuation. Explicit control laws have been proposed for both parabolic and hyperbolic PDEs (e.g., Krstic & Smyshlyaev, 2008 and references therein). However, control aspects of PDEs via diffusivity actuator have been seldom discussed (e.g., Gao, 1999). Diffusivity control, as a special type of bilinear control, can improve the controllability obtained by just using either interior or boundary control (see, e.g., Lin et al., 2006).

In this paper we consider an optimal control problem for a parabolic system with diffusivity and interior actuation mechanisms. This problem arises in the control of the toroidal current density radial profile in magnetically confined fusion plasmas within tokamak reactors (Pironti & Walker, 2005). The achievement of a suitable toroidal current profile plays an important role in enabling high fusion gain and non-inductive sustainment of the plasma current for steady-state operation (see, e.g., Taylor et al.,

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* Corresponding address: Institute of Cyber-Systems & Control, Zhejiang University, Hangzhou, Zhejiang 310027, China. Tel.: +86 57187952233x1165; fax: +86 571 87952279.

E-mail address: cxu@csc.zju.edu.cn (C. Xu).

1997). In a tokamak, it is possible to use the poloidal component B_{pol} of the helicoidal magnetic lines confining the plasma to define nested toroidal surfaces corresponding to constant values of the poloidal magnetic flux. The poloidal flux ψ at a point P is the total flux through the surface S bounded by the toroidal ring passing through P , i.e., $\psi = \int B_{pol} dS$. The dynamics of the poloidal flux ψ is governed in normalized cylindrical coordinates by a nonlinear parabolic partial differential equation (PDE) usually referred to as the magnetic flux diffusion equation. By exploiting the time scale separation in the evolution of the kinetic and magnetic variables (Ouet et al., 2007), this equation can be written as

$$\frac{\partial \psi}{\partial t} = f_D(\hat{\rho})u_D(t) \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} \left(\hat{\rho} f(\hat{\rho}) \frac{\partial \psi}{\partial \hat{\rho}} \right) + f_I(\hat{\rho})u_I(t), \quad (1)$$

with boundary conditions

$$\left. \frac{\partial \psi}{\partial \hat{\rho}} \right|_{\hat{\rho}=0} = 0, \quad \left. \frac{\partial \psi}{\partial \hat{\rho}} \right|_{\hat{\rho}=1} = u_B(t), \quad (2)$$

where $u_D(t)$, $u_I(t)$, $u_B(t)$ denote the diffusivity, interior and boundary control inputs respectively, f , f_D , f_I are spatial functions, and $\hat{\rho}$ represents the normalized spatial coordinate indexing the magnetic flux surfaces. The evolution in time of the current profile is related to the evolution of the spatial derivative of the poloidal flux profile. Therefore, we can control the current profile by controlling the shape of the poloidal flux profile. We are interested in exploiting the availability of physical actuators regulating inductive plasma current, line-averaged density and non-inductive current drive power to steer the plasma poloidal flux to a desired profile in a designated time period, which defines a finite-time optimal control problem. By modulating these physical actuators it is possible not only to vary the amount of non-inductive current driven into the system (interior control $u_I(t)$) and the total plasma current (boundary control $u_B(t)$) but also to modify the electrical resistivity of the plasma (diffusivity control $u_D(t)$).

Another example of PDE systems with the potential of bilinear control (not necessarily diffusivity control) can be found in the areas of flow control (Lenhart, 1995). In Lenhart (1995), a saturated flow through a one-dimensional idealized tube packed with soil is considered. The soil contains contaminant samples and a fluid is pumped through the tube (from left to right) to remove the contaminants. The velocity of the fluid pumped into the tube is considered as the control variable which appears as the convective coefficient in the convective–diffusive PDE system governing the contaminant concentration.

The design of optimal control strategies, particularly in closed-loop, for an infinite dimensional system is often numerically unfeasible. In this case, reduced order modeling techniques may become crucial. In this paper, we use the proper orthogonal decomposition (POD) method to obtain a low dimensional dynamical system (LDDS) for the parabolic PDE. The POD method is an efficient reduced order modeling (ROM) technique used to obtain LDDS's from data ensembles which arise in numerical simulation or experimental observation. The POD method has been widely used and proved successful to discover coherent structures from complex physical processes (see, e.g., Kunisch & Volkwein, 1999). In Kunisch and Volkwein (1999), the POD approach is applied to derive a reduced order model of the Burgers' equation, and then the associated optimal control is considered by using the sequential quadratic programming (SQP) method. Fundamental aspects of POD methods applied to parabolic problems, such as error estimates of Galerkin–POD for both linear and nonlinear parabolic systems, are discussed in Kunisch and Volkwein (2001).

The obtained reduced order system in this work is a bilinear system. For the solution of bilinear optimal control problems,

several convergent schemes based on quasi-linearization has been proposed (Aganovic & Gajic, 1994; Banks & Dinesh, 2000; Cimen & Banks, 2004; Hofer & Tibken, 1988). Convergence properties of methods based on quasi-linearization of the system dynamics (Banks & Dinesh, 2000; Cimen & Banks, 2004) and of the optimality conditions (Aganovic & Gajic, 1994; Hofer & Tibken, 1988) are compared in this work. Based on these convergence properties, we focus on the latter group, which guarantees convergence to the solution of the two boundary value problem derived from Pontryagin's principle. The algorithm in Hofer and Tibken (1988), constructs linear systems by updating system and input matrices at each iteration step. The linear state–costate duality structure of the optimality conditions is preserved at each iteration step. Then, Riccati equations are derived to establish successive feedback laws. Similarly, instead of solving a Riccati equation iteratively, a Lyapunov equation is solved at each iteration step in Aganovic and Gajic (1994). Inspired by Aganovic and Gajic (1994), Hofer and Tibken (1988) and the work in Tang (2005) for nonlinear systems with linear control, we present an iterative scheme based on the optimality condition, which introduces an inhomogeneous term in the successive linear state–costate duality structure. In comparison to previous results for bilinear systems, the proposed scheme avoids repeated computations of the Riccati (or Lyapunov) equation at each iteration step by introducing an iterative scheme for the inhomogeneous term involved in the feedback law. In terms of the number of ODEs required to solve the Riccati (or Lyapunov) matrix equation and the inhomogeneous term equation, the proposed method can decrease the number of ODEs to be computed at each iteration step from l^2 to l , where l denotes the system dimension. The convergence proof, based on contraction mapping theory (Khalil, 2002), tackles unique characteristics of the proposed approach.

Motivated by the current profile control problem in tokamak plasmas, this paper represents a new effort to connect, by using model reduction, nonlinear parabolic PDE optimal feedback control and iterative optimal control methodologies for finite dimensional systems. The paper is organized as follows. The optimal control problem is stated in Section 2. In Section 3, we discuss the POD method to obtain reduced order models. In Section 4, the Galerkin projection method is used to obtain a reduced order model based on a test function set composed by dominant POD modes. In Section 5, we propose an iterative convergent method based on successive approximations to compute the optimal controls. Simulation studies are presented in Section 6. Section 7 closes the paper by stating the conclusions.

2. Problem statement

We consider a 1D parabolic system over $\Omega = \{(x, t) : 0 \leq x \leq L, t_0 \leq t \leq t_f\}$, which is governed by

$$\frac{\partial z}{\partial t} = (1 + u_D(t))\gamma(x) \frac{\partial}{\partial x} \left(\zeta(x) \frac{\partial z}{\partial x} \right) + \lambda(x)z + \xi(x)u_I(t), \quad (3)$$

$$\frac{\partial z}{\partial x}(0, t) = \frac{\partial z}{\partial x}(L, t) = 0, \quad z(x, t_0) = \varphi(x), \quad (4)$$

where $z(x, t)$ represents the system state, $u_D(t)$ and $u_I(t)$ the diffusivity and interior controls respectively, and $\varphi(x)$ the initial distribution. We assume that $\gamma(x)$, $\zeta(x)$, $\lambda(x)$ and $\xi(x)$ are continuous functions ($\gamma(x)$ and $\zeta(x)$ positive). Note that (1) and (2) can be written as in (3) and (4). The state $z(x, t)$ in (3) and (4) may represent the error between actual and desired poloidal flux profiles. In this work we consider $u_B(t) = 0$, i.e., no boundary actuation.

Remark 1 (Scalar Transformation). If the time varying function $u_D(t)$ were known, it would be possible to consider a scalar transformation $\tau = t + \int_0^t u_D(s)ds$ (where $u_D(t)$ is assumed to be more than -1 to ensure well-posedness), rewrite system (3) as $\frac{\partial z}{\partial \tau} = \gamma(x) \frac{\partial}{\partial x} (\zeta(x) \frac{\partial z}{\partial x}) + \lambda(x)z u(t) + \xi(x)v(t)$ with $u(t) = \frac{1}{1+u_D(t)}$ and $v(t) = \frac{u_i(t)}{1+u_D(t)}$, and remove in this way the diffusivity control from the elliptic operator term. However, in this work the diffusivity control $u_D(t)$ is unknown and to be designed. Therefore, the new time scale τ is also unknown, which makes the design of any finite-time optimal controller very challenging. The scalar transformation does not eliminate then the difficulties associated with the to-be-designed diffusivity control.

We state an optimal control problem for the parabolic system (3) with the following cost functional

$$\begin{aligned} \min_{u_D, u_i} \mathcal{J} = & \frac{1}{2} \int_0^L \mathcal{S}(x)z^2(x, t_f)dx + \frac{1}{2} \int_{\Omega} \mathcal{Q}(x)z^2(x, t)dxdt \\ & + \frac{1}{2} \int_0^{t_f} (r_I u_i^2 + r_D u_D^2)dt, \end{aligned} \quad (5)$$

where $\mathcal{S}(x)$ and $\mathcal{Q}(x)$ are positive weight functions; r_I and r_D are positive definite control weighting factors.

In Xu et al. (2008), we have demonstrated the existence of solution for this optimal control problem, and obtained open-loop controllers using the Sequential Quadratic Programming (SQP) optimization algorithm. However, the uniqueness of optimal control solution of an arbitrary bilinear infinite dimensional system cannot be guaranteed in general because of the convexity limitations due to the bilinearity of the problem. Uniqueness of solution can only be proved under special conditions. For instance, in Addou and Benbrik (2002) the authors have proved uniqueness of solution for the optimal control problem of a bilinear distributed parameter system only when the initial state satisfies specific smallness conditions.

3. POD reduced order modeling

The set $\mathcal{V} = \text{span}\{z_1, \dots, z_n\} \subset \mathbb{R}^m$ refers to a data ensemble consisting of the snapshots $\{z_j\}_{j=1}^n$, obtained from simulation or experimental observation, of the values of the state $z(x, t)$ at m points in space at n instants in time. Let $\{\bar{\psi}_k\}_{k=1}^d$ be an orthonormal basis of the data ensemble \mathcal{V} , where $d = \dim \mathcal{V} \leq m$. We then project each of the snapshots onto the basis $\bar{\psi}_k$, $z_j = \sum_{k=1}^d (z_j^T \bar{\psi}_k) \bar{\psi}_k$, $j = 1, \dots, n$. The goal of the POD method is to find an orthonormal basis such that for some predefined l , $1 \leq l \leq d$ the following average index is minimized

$$\begin{aligned} \min_{\{\bar{\psi}_k\}_{k=1}^l} & \frac{1}{n} \sum_{j=1}^n \left\| z_j - \sum_{k=1}^l (z_j^T \bar{\psi}_k) \bar{\psi}_k \right\|^2, \\ \text{subject to} & \quad (\bar{\psi}_i^T \bar{\psi}_j) = \delta_{ij}, \quad 1 \leq i \leq l, \quad 1 \leq j \leq l, \\ \text{where } \|z\| & = \sqrt{z^T z}, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \end{aligned} \quad (6)$$

Defining the correlation matrix $K \in \mathbb{R}^{n \times n}$ as $K_{ij} = \frac{1}{n} (z_j^T z_i)$, for $i, j = 1, \dots, n$, the following singular value decomposition result (Kunisch & Volkwein, 2001) follows:

Theorem 1. Let $\lambda_1 > \dots > \lambda_l > \dots > \lambda_d > 0$ denote the positive eigenvalues of the correlation matrix K and $v_1, \dots, v_l, \dots, v_d$ the associated eigenvectors, where $d = \text{rank}(K)$. Then, the POD basis functions are given by

$$\bar{\psi}_k = \frac{1}{\sqrt{\lambda_k}} \sum_{j=1}^n (v_k)_j z_j = \frac{1}{\sqrt{\lambda_k}} Z v_k, \quad (k = 1, \dots, d), \quad (7)$$

where $(v_k)_j$ is the j th component of the eigenvector v_k and $Z = (z_1, \dots, z_n)$ is the collection of all the snapshots. Moreover, the error (energy ratio) associated with the approximation with the first l POD modes is

$$\varepsilon_l = \frac{1}{n} \sum_{j=1}^n \left\| z_j - \sum_{k=1}^l (z_j^T \bar{\psi}_k) \bar{\psi}_k \right\|^2 = \sum_{k=l+1}^d \lambda_k. \quad (8)$$

4. POD/Galerkin method

We let $\psi_j(x) \in V_{\text{POD}}$ be a test function, where $V_{\text{POD}} = \text{span}\{\psi_1(x), \dots, \psi_l(x)\}$ is the test function space, and $\bar{\psi}_k(x)$ ($k = 1, \dots, l$) is obtained as an interpolation of the POD vector $\bar{\psi}_k$ ($k = 1, \dots, l$). Then, we multiply both sides of (3) by the test function $\psi_j(x) \in V_{\text{POD}}$, for $j = 1, \dots, l$, and integrate by parts taking into account that $\frac{\partial \psi_j}{\partial x}(0) = \frac{\partial \psi_j}{\partial x}(L) = 0$, to obtain the following weak form

$$\begin{aligned} \int_0^L \frac{\partial z}{\partial t} \psi_j(x) dx + \int_0^L (1 + u_D(t)) \zeta(x) \frac{\partial z}{\partial x} \frac{\partial (\gamma(x) \psi_j(x))}{\partial x} dx \\ = \int_0^L \xi(x) u_i(t) \psi_j(x) dx + \int_0^L \lambda(x) z \psi_j(x) dx. \end{aligned} \quad (9)$$

We implement the Galerkin approximation $z(x, t) \approx y(x, t) = \sum_{k=1}^l \alpha_k(t) \psi_k(x)$, and substitute this expression for $z(x, t)$ into the weak form (9). Taking into account that $\int_0^L \psi_j(x) \psi_k(x) dx = \delta_{jk}$ (see, (6)), we can obtain the following finite dimensional system:

$$\frac{dy}{dt} = Ay + Ky u_D(t) + Fu_i(t), \quad (10)$$

where $A = K + G$ and $K_{jk} = -\int_0^L \zeta \frac{\partial (\gamma \psi_j)}{\partial x} \frac{\partial \psi_k}{\partial x} dx$, $G_{jk} = \int_0^L \lambda(x) \psi_j(x) \psi_k(x) dx$, $F_j = \int_0^L \xi(x) \psi_j(x) dx$, with $y(t) = [\alpha_1(t), \dots, \alpha_l(t)]^T \in \mathbb{R}^l$, $G, K, A \in \mathbb{R}^{l \times l}$, $F \in \mathbb{R}^l$. The vector $y(t)$ is the finite dimensional approximation, with respect to the obtained POD modes, of the variable $z(x, t)$ in (3). The initial values are given by $\alpha_j(0) = \int_0^L z(x, 0) \psi_j(x) dx$, $j = 1, 2, \dots, l$.

5. Bilinear quadratic optimal control

The finite horizon optimal control problem defined in (5) can now be rewritten as

$$\begin{aligned} \min_{u_i, u_D} J = & \frac{1}{2} [y(t_f)]^T S [y(t_f)] + \frac{1}{2} \int_{t_0}^{t_f} y^T(t) Q y(t) dt \\ & + \frac{1}{2} \int_{t_0}^{t_f} (r_I u_i^2(t) + r_D u_D^2(t)) dt, \end{aligned} \quad (11)$$

where the elements of S and Q are defined by $S_{ij} = \int_0^L \mathcal{S}(x) \psi_i(x) \psi_j(x) dx$ and $Q_{ij} = \int_0^L \mathcal{Q}(x) \psi_i(x) \psi_j(x) dx$ respectively, for $i, j = 1, \dots, l$.

Introducing the Lagrange multiplier $p \in \mathbb{R}^l$, we can define the system Hamiltonian

$$\begin{aligned} \mathcal{H}(y, u_i, u_D, p) = & \frac{1}{2} y^T Q y + \frac{1}{2} r_I u_i^2 + \frac{1}{2} r_D u_D^2 \\ & + p^T (Ay + Ky u_D + Fu_i). \end{aligned} \quad (12)$$

The minimizing control laws are given by

$$\frac{\partial \mathcal{H}}{\partial u_i} = 0 \Rightarrow u_i = -r_I^{-1} F^T p, \quad (13)$$

$$\frac{\partial \mathcal{H}}{\partial u_D} = 0 \Rightarrow u_D = -r_D^{-1} (Ky)^T p. \quad (14)$$

Thus, using the maximum principle, a canonical optimality condition can be obtained,

$$\begin{cases} \dot{y} = \frac{\partial \mathcal{H}}{\partial p} = Ay - Ky r_D^{-1} (Ky)^T p - Fr_I^{-1} F^T p, \\ \dot{p} = -\frac{\partial \mathcal{H}}{\partial y} = -Qy - A^T p + K^T p r_D^{-1} (Ky)^T p, \end{cases} \quad (15)$$

with $y(t_0) = y_0$ and $p(t_f) = Sy(t_f)$, which is a nonlinear two-point boundary value problem (TBVP) and usually impossible to be solved explicitly. The problem of existence and uniqueness of the solutions of (15) has been studied extensively (see, e.g., Tisdell, 2006 and references therein).

5.1. Quasi-linearization of system dynamics

By defining the sequence of system approximations (Banks & Dinesh, 2000; Cimen & Banks, 2004)

$$\dot{y}^{(k+1)} = Ay^{(k+1)} + Ky^{(k)} u_D^{(k+1)} + Fu_I^{(k+1)} \quad (16)$$

with the cost functional at each iteration step given by

$$J^{(k+1)} = \frac{1}{2} [y^{(k+1)}(t_f)]^T S y^{(k+1)}(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \{ [y^{(k+1)}]^T Q y^{(k+1)} + r_I [u_I^{(k+1)}]^2 + r_D [u_D^{(k+1)}]^2 \} dt, \quad (17)$$

we can introduce the Lagrange multiplier $p^{(k+1)} \in \mathbb{R}^l$ and define the system Hamiltonian

$$\begin{aligned} \mathcal{H}^{(k+1)}(y^{(k+1)}, u_I^{(k+1)}, u_D^{(k+1)}, p^{(k+1)}) \\ = \frac{1}{2} \{ [y^{(k+1)}]^T Q y^{(k+1)} + r_I [u_I^{(k+1)}]^2 + r_D [u_D^{(k+1)}]^2 \} \\ + [p^{(k+1)}]^T [Ay^{(k+1)} + Fu_I^{(k+1)} + Ky^{(k)} u_D^{(k+1)}]. \end{aligned}$$

The minimizing control laws are then given by

$$\frac{\partial \mathcal{H}^{(k+1)}}{\partial u_I^{(k+1)}} = 0 \Rightarrow u_I^{(k+1)}(t) = -r_I^{-1} F^T p^{(k+1)}, \quad (18)$$

$$\frac{\partial \mathcal{H}^{(k+1)}}{\partial u_D^{(k+1)}} = 0 \Rightarrow u_D^{(k+1)}(t) = -r_D^{-1} [Ky^{(k)}]^T p^{(k+1)}, \quad (19)$$

and the canonical equations become

$$\begin{cases} \dot{y}^{(k+1)} = Ay^{(k+1)} - (U^{(k)} + V)p^{(k+1)}, \\ \dot{p}^{(k+1)} = -Qy^{(k+1)} - A^T p^{(k+1)}, \end{cases} \quad (20)$$

with $y^{(k+1)}(t_0) = y_0$ and $p^{(k+1)}(t_f) = Sy^{(k+1)}(t_f)$, where $U^{(k)} = Ky^{(k)} r_D^{-1} [Ky^{(k)}]^T$ and $V = Fr_I^{-1} F^T = V^T$. At each iteration step we assume

$$p^{(k+1)} = P^{(k+1)} y^{(k+1)}, \quad (21)$$

to obtain the Riccati equation

$$\begin{aligned} \dot{P}^{(k+1)} = -P^{(k+1)} A - A^T P^{(k+1)} + P^{(k+1)} W^{(k)} P^{(k+1)} - Q, \\ P^{(k+1)}(t_f) = S, \end{aligned} \quad (22)$$

where $W^{(k)} = U^{(k)} + V$. Once convergence is achieved, i.e., $\lim_{k \rightarrow \infty} P^{(k)} = P^*$, the suboptimal control laws can be written as

$$u_I^* = -r_I^{-1} F^T P^* y, \quad u_D^* = -r_D^{-1} (Ky)^T P^* y. \quad (23)$$

Remark 2. This technique is an intuitive iteration scheme obtained by rewriting nonlinear systems into quasi-linear systems, where the system matrices depend nonlinearly on the state variable. Under certain conditions, this iteration scheme can achieve convergence (Banks & Dinesh, 2000; Cimen & Banks, 2004). However, the relation between the convergent solution and the optimality condition (Pontryagin maximum principle) is not clear. When (15) and (20) are compared, the difference in the \dot{p} equation is evident.

5.2. Quasi-linearization of optimality conditions

To ensure convergence to a solution of the TBVP (15), we solve the optimal control problem defined for the bilinear system (10) by proposing a Picard-approximation-based successive approach for the TBVP (15) itself, i.e.,

$$\begin{cases} \dot{y}^{(k+1)} = Ay^{(k+1)} - V p^{(k+1)} - G^{(k)}, \\ \dot{p}^{(k+1)} = -Qy^{(k+1)} - A^T p^{(k+1)} + H^{(k)}, \end{cases} \quad (24)$$

with $y^{(k+1)}(t_0) = y_0$ and $p^{(k+1)}(t_f) = Sy^{(k+1)}(t_f)$, where $V = Fr_I^{-1} F^T = V^T$, $G^{(k)} = Ky^{(k)} r_D^{-1} [Ky^{(k)}]^T p^{(k)}$ and $H^{(k)} = r_D^{-1} [Ky^{(k)}]^T p^{(k)} K^T p^{(k)}$. The minimizing control laws (13) and (14) are given by

$$u_I^{(k+1)}(t) = -r_I^{-1} F^T p^{(k+1)}, \quad (25)$$

$$u_D^{(k+1)}(t) = -r_D^{-1} [Ky^{(k+1)}]^T p^{(k+1)}. \quad (26)$$

To solve the linear two boundary value problem (24), it is standard to assume

$$p^{(k+1)} = P y^{(k+1)} + q^{(k+1)}, \quad P^T = P, \quad (27)$$

in order to obtain

$$\begin{aligned} \dot{P} = -PA - A^T P + PVP - Q, \quad P(t_f) = S, \\ \dot{q}^{(k+1)} = -(A - VP)^T q^{(k+1)} + PG^{(k)} + H^{(k)}, \\ q^{(k+1)}(t_f) = 0, \end{aligned} \quad (28)$$

where $G^{(k)} = r_D^{-1} Ky^{(k)} [Ky^{(k)}]^T [Py^{(k)} + q^{(k)}]$, $H^{(k)} = r_D^{-1} [Ky^{(k)}]^T [Py^{(k)} + q^{(k)}] K^T [Py^{(k)} + q^{(k)}]$. Then, at each iteration step, the closed-loop system becomes

$$\begin{aligned} \dot{y}^{(k+1)} = (A - VP)y^{(k+1)} - Vq^{(k+1)} - G^{(k)}, \\ y^{(k+1)}(t_0) = y_0. \end{aligned} \quad (29)$$

When the iteration index $k \rightarrow \infty$, we obtain the following feedback laws,

$$u_I = -r_I^{-1} F^T (Py + q^*), \quad (30)$$

$$u_D = -r_D^{-1} (Ky)^T (Py + q^*), \quad (31)$$

where $q^* = \lim_{k \rightarrow \infty} q^{(k)}$.

Remark 3. When $\lim_{k \rightarrow \infty} y^{(k)} = y^*$ and $\lim_{k \rightarrow \infty} q^{(k)} = q^*$, we have $y^{(k)} - y^{(k+1)} \rightarrow 0$ and $q^{(k)} - q^{(k+1)} \rightarrow 0$. Therefore, the TBVP (24) reduces to the TBVP (15), and the fixed point (y^*, q^*) of the iteration scheme solves (15).

Remark 4. The solution of the Riccati matrix equation (P -equation) actually requires the solution of l^2 coupled ODEs, where l denotes the system dimension. The advantage of this algorithm resides on the fact that it is not necessary to compute the Riccati equation in each iteration step. Only the vector equation for the feed-forward control term (q -equation) needs to be solved iteratively in each step. However, the solution of this equation requires the solution of only l coupled ODEs.

5.2.1. Convergence proof

In the rest of this section, it remains to prove the convergence of the proposed Picard approximation in solving the optimal control problem. Namely, we will show the following limits in appropriate functional spaces, i.e., $\lim_{k \rightarrow \infty} y^{(k)} = y^*$ and $\lim_{k \rightarrow \infty} q^{(k)} = q^*$. The associated spaces are two Banach spaces (see, e.g., page 76 of Conway, 1994), $\mathfrak{B}_1 = \mathfrak{B}_2 = C([t_0, t_f], \mathbb{R}^l)$, with norms $\|y\|_{\mathfrak{B}_1} = \sup_{s \in [t_0, t_f]} \|y(s)\|$, for any $y \in \mathfrak{B}_1$ and $\|q\|_{\mathfrak{B}_2} = \sup_{s \in [t_0, t_f]} \|q(s)\|$,

for any $q \in \mathfrak{B}_2$, where $\|y\| = \sqrt{\sum_{i=1}^l y_i^2}$, and $\|q\| = \sqrt{\sum_{i=1}^l q_i^2}$. We define a matrix norm based on the vector norm $\|\cdot\|$, i.e., $\|A\| = \sup_{y \neq 0} \frac{\|Ay\|}{\|y\|}$ which can be computed via $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$ (see, i.e., page 57 of Golub & Loan, 1996).

Remark 5. To show $\lim_{k \rightarrow \infty} y^{(k)} = y^*$ and $\lim_{k \rightarrow \infty} q^{(k)} = q^*$, we only need to show that both $\{y^{(k)}\}$ and $\{q^{(k)}\}$ are Cauchy sequences. Thus, the convergence follows due to the completeness of the Banach spaces. The convergence proof is based on the contraction mapping theorem for Banach spaces (Khalil, 2002).

Based on (28) and (29), we obtain differential equations for the differences $q^{(k+1)} - q^{(k)}$ and $y^{(k+1)} - y^{(k)}$, i.e.,

$$\frac{d}{dt}[y^{(k+1)} - y^{(k)}] = (A - VP)[y^{(k+1)} - y^{(k)}] - V[q^{(k+1)} - q^{(k)}] - [G^{(k)} - G^{(k-1)}], \quad (32)$$

$$\frac{d}{dt}[q^{(k+1)} - q^{(k)}] = -(A - VP)^T[q^{(k+1)} - q^{(k)}] + P[G^{(k)} - G^{(k-1)}] + [H^{(k)} - H^{(k-1)}]. \quad (33)$$

We integrate (32) over (t_0, t) and take into account $y^{(k+1)}(t_0) = y^{(k)}(t_0)$ to obtain

$$y^{(k+1)} - y^{(k)} = - \int_{t_0}^t e^{(A-VP)(t-\tau)} V[q^{(k+1)} - q^{(k)}] d\tau - \int_{t_0}^t e^{(A-VP)(t-\tau)} [G^{(k)} - G^{(k-1)}] d\tau. \quad (34)$$

To integrate (33) with the final value $q^{(k+1)}(t_f) - q^{(k)}(t_f)$ given, we multiply both sides of (33) by $e^{(A-VP)^T t}$ to obtain the following equality

$$\frac{d}{dt}\{e^{(A-VP)^T t}[q^{(k+1)} - q^{(k)}]\} = e^{(A-VP)^T t}\{P[G^{(k)} - G^{(k-1)}] + [H^{(k)} - H^{(k-1)}]\}. \quad (35)$$

Then, we integrate both sides of (35) from t to t_f to obtain

$$e^{(A-VP)^T t_f}[q^{(k+1)}(t_f) - q^{(k)}(t_f)] - e^{(A-VP)^T t}[q^{(k+1)}(t) - q^{(k)}(t)] = \int_t^{t_f} e^{(A-VP)^T \tau} P[G^{(k)}(\tau) - G^{(k-1)}(\tau)] d\tau + \int_t^{t_f} e^{(A-VP)^T \tau} [H^{(k)}(\tau) - H^{(k-1)}(\tau)] d\tau. \quad (36)$$

By multiplying both sides of the integral (36) by $e^{-(A-VP)^T t}$, we can obtain

$$q^{(k+1)}(t) - q^{(k)}(t) = - \int_t^{t_f} e^{-(A-VP)^T(t-\tau)} P[G^{(k)}(\tau) - G^{(k-1)}(\tau)] d\tau - \int_t^{t_f} e^{-(A-VP)^T(t-\tau)} [H^{(k)}(\tau) - H^{(k-1)}(\tau)] d\tau, \quad (37)$$

where we have taken into account the final value given in (28), i.e., $q^{(k+1)}(t_f) = q^{(k)}(t_f) = 0$.

Theorem 2. If the control weighting factor r_D is large enough, then the iteration scheme is convergent, i.e., $\lim_{k \rightarrow \infty} y^{(k)} = y^*$, $\lim_{k \rightarrow \infty} q^{(k)} = q^*$.

Proof. Based on (34) and (37), we compute the norms,

$$\|y^{(k+1)} - y^{(k)}\| \leq \int_{t_0}^{t_f} \gamma_1 \|q^{(k+1)} - q^{(k)}\| d\tau + \int_{t_0}^{t_f} \gamma_2 \|G^{(k)} - G^{(k-1)}\| d\tau, \quad (38)$$

$$\|q^{(k+1)} - q^{(k)}\| \leq \int_{t_0}^{t_f} \gamma_3 \|G^{(k)} - G^{(k-1)}\| d\tau + \int_{t_0}^{t_f} \gamma_4 \|H^{(k)} - H^{(k-1)}\| d\tau, \quad (39)$$

where the time varying γ -coefficients are $\gamma_1(\tau) = \|e^{(A-VP)(t_f-\tau)} V\|$, $\gamma_2(\tau) = \|e^{(A-VP)(t_f-\tau)}\|$, $\gamma_3(\tau) = \|e^{-(A-VP)^T(t_0-\tau)} P\|$ and $\gamma_4(\tau) = \|e^{-(A-VP)^T(t_0-\tau)}\| = \gamma_2(t_f + t_0 - \tau)$.

We rewrite

$$G^{(k)} = r_D^{-1} K Y^{(k)} K^T [P y^{(k)} + q^{(k)}], \quad (40)$$

$$H^{(k)} = r_D^{-1} \delta_{yy}^{(k)} K^T [P y^{(k)} + q^{(k)}] + r_D^{-1} \delta_{yq}^{(k)} K^T [P y^{(k)} + q^{(k)}], \quad (41)$$

where $Y^{(k)} = y^{(k)} [y^{(k)}]^T$, $\delta_{yy}^{(k)} = [y^{(k)}]^T K^T P y^{(k)}$ and $\delta_{yq}^{(k)} = [y^{(k)}]^T K^T q^{(k)}$. Now we evaluate $G^{(k)} - G^{(k-1)}$ and $H^{(k)} - H^{(k-1)}$ appearing in (38) and (39) in terms of $q^{(k)} - q^{(k-1)}$ and $y^{(k)} - y^{(k-1)}$,

$$\frac{\|G^{(k)} - G^{(k-1)}\|}{r_D^{-1}} \leq \|K[Y^{(k)} - Y^{(k-1)}]K^T[P y^{(k)} + q^{(k)}]\| + \|K Y^{(k-1)} K^T P[y^{(k)} - y^{(k-1)}]\| + \|K Y^{(k-1)} K^T [q^{(k)} - q^{(k-1)}]\|, \quad (42)$$

where

$$Y^{(k)} - Y^{(k-1)} = [y^{(k)} - y^{(k-1)}][y^{(k)}]^T + y^{(k-1)}[y^{(k)} - y^{(k-1)}]^T. \quad (43)$$

Then, we have

$$\frac{\|G^{(k)} - G^{(k-1)}\|}{r_D^{-1}} \leq \gamma_5^{(k)} \|y^{(k)} - y^{(k-1)}\| + \gamma_6^{(k)} \|q^{(k)} - q^{(k-1)}\|, \quad (44)$$

where $\gamma_5^{(k)}(\tau) = [\|y^{(k)}\| + \|y^{(k-1)}\|] \|K\|^2 \|P y^{(k)} + q^{(k)}\| + \|K\|^2 \|P\| \|Y^{(k-1)}\|$ and $\gamma_6^{(k)}(\tau) = \|K\|^2 \|Y^{(k-1)}\|$. Similarly, we have

$$\frac{\|H^{(k)} - H^{(k-1)}\|}{r_D^{-1}} \leq |\delta_{yy}^{(k)} - \delta_{yy}^{(k-1)}| + |\delta_{yq}^{(k)} - \delta_{yq}^{(k-1)}| \|K^T\| \|P y^{(k)} + q^{(k)}\| + [|\delta_{yy}^{(k-1)}| + |\delta_{yq}^{(k-1)}|] \|K^T P\| \|y^{(k)} - y^{(k-1)}\| + [|\delta_{yy}^{(k-1)}| + |\delta_{yq}^{(k-1)}|] \|K^T\| \|q^{(k)} - q^{(k-1)}\|. \quad (45)$$

Noting

$$\delta_{yy}^{(k)} - \delta_{yy}^{(k-1)} = [y^{(k)} - y^{(k-1)}]^T K^T P y^{(k)} + [y^{(k-1)}]^T K^T P [y^{(k)} - y^{(k-1)}] \quad (46)$$

$$\delta_{yq}^{(k)} - \delta_{yq}^{(k-1)} = [y^{(k)} - y^{(k-1)}]^T K^T q^{(k)} + [y^{(k-1)}]^T K^T [q^{(k)} - q^{(k-1)}] \quad (47)$$

then we can obtain

$$\frac{\|H^{(k)} - H^{(k-1)}\|}{r_D^{-1}} \leq \gamma_7^{(k)} \|y^{(k)} - y^{(k-1)}\| + \gamma_8^{(k)} \|q^{(k)} - q^{(k-1)}\|, \quad (48)$$

where

$$\begin{aligned} \gamma_7^{(k)}(\tau) = & \|K^T\| \|Py^{(k)} + q^{(k)}\| \|K^T P\| (\|y^{(k)}\| + \|y^{(k-1)}\|) \\ & + [|\delta_{yy}^{(k-1)}| + |\delta_{yq}^{(k-1)}|] \|K^T P\| \\ & + \|K^T\| \|Py^{(k)} + q^{(k)}\| \|K^T q^{(k)}\|, \end{aligned} \quad (49)$$

$$\begin{aligned} \gamma_8^{(k)}(\tau) = & \|K^T\| \|Py^{(k)} + q^{(k)}\| \|K\| \|y^{(k-1)}\| \\ & + [|\delta_{yy}^{(k-1)}| + |\delta_{yq}^{(k-1)}|] \|K^T\|. \end{aligned} \quad (50)$$

Therefore, by computing \mathfrak{B} -norms for both sides of (38) and (39), we obtain

$$\begin{bmatrix} \|y^{(k+1)} - y^{(k)}\|_{\mathfrak{B}_1} \\ \|q^{(k+1)} - q^{(k)}\|_{\mathfrak{B}_2} \end{bmatrix} \leq \frac{T^{(k)}}{r_D} \begin{bmatrix} \|y^{(k)} - y^{(k-1)}\|_{\mathfrak{B}_1} \\ \|q^{(k)} - q^{(k-1)}\|_{\mathfrak{B}_2} \end{bmatrix} \quad (51)$$

where the elements of the transform matrix T are given by

$$\begin{aligned} T_{11}^{(k)} &= (t_f - t_0) \max_{\tau \in [t_0, t_f]} [\gamma_2(\tau) \gamma_5^{(k)}(\tau)], \\ T_{12}^{(k)} &= (t_f - t_0) \max_{\tau \in [t_0, t_f]} [\gamma_1(\tau) + \gamma_2(\tau) \gamma_6^{(k)}(\tau)], \\ T_{21}^{(k)} &= (t_f - t_0) \max_{\tau \in [t_0, t_f]} [\gamma_3(\tau) \gamma_5^{(k)}(\tau) + \gamma_4(\tau) \gamma_7^{(k)}(\tau)], \\ T_{22}^{(k)} &= (t_f - t_0) \max_{\tau \in [t_0, t_f]} [\gamma_3(\tau) \gamma_6^{(k)}(\tau) + \gamma_4(\tau) \gamma_8^{(k)}(\tau)]. \end{aligned}$$

Therefore, if all of the eigenvalues of $T^{(k)}$, $\sigma(T^{(k)})$ satisfy $\frac{t_f - t_0}{r_D} \max |\sigma(T^{(k)})| < 1$, then we can conclude that the sequences $\{y^{(k)}\}$ and $\{q^{(k)}\}$ are Cauchy.

The transformation matrix $T^{(k)}$ calculated by $T_{11}^{(k)}, T_{12}^{(k)}, T_{21}^{(k)}$ and $T_{22}^{(k)}$ (above) is dependent on the iteration index k and also includes the evolutions of $y^{(k)}$ and $q^{(k)}$. Therefore, it is difficult and almost impossible to compute the eigenvalues of $T^{(k)}$ explicitly in each iteration step. However, to ensure the convergence of the iteration scheme it is possible to make the control weighting factor r_D large enough since this is also a sufficient condition for $T^{(k)}$ to be uniformly bounded in terms of the iteration index k , as it will be proved below. Making r_D large enough is also a way to ensure $|u_D| < 1$.

Since for the 2×2 matrices $T^{(k)}$ all norms are equivalent, uniform boundedness of $T^{(k)}$ requires all elements of $T^{(k)}$ uniformly bounded in k . We note that the elements of $T^{(k)}$ are expressed in terms of the iteration constants $\gamma_5^{(k)}, \gamma_6^{(k)}, \gamma_7^{(k)}$ and $\gamma_8^{(k)}$, which in turns are defined in terms of the norms of the iterative sequences $\{y^{(k)}\}$ and $\{q^{(k)}\}$. Therefore, we need to show that the iterative sequences $\{y^{(k)}\}$ and $\{q^{(k)}\}$ are uniformly bounded in k . Based on (28) and (29), we can write the solutions for $y^{(k+1)}$ and $q^{(k+1)}$ as

$$\begin{aligned} y^{(k+1)}(t) = & e^{(A-VP)(t-t_0)} y^{(k+1)}(t_0) \\ & - \int_{t_0}^t e^{(A-VP)(t-\tau)} \left[V \int_{\tau}^{t_f} e^{(A-VP)^T(\tau-\mu)} [PG^{(k)}(\mu) \right. \\ & \left. + H^{(k)}(\mu)] d\mu + G^{(k)}(\tau) \right] d\tau, \end{aligned} \quad (52)$$

$$q^{(k+1)}(t) = - \int_t^{t_f} e^{(A-VP)^T(t-\tau)} [PG^{(k)}(\tau) + H^{(k)}(\tau)] d\tau. \quad (53)$$

We note that $G^{(k)}$ and $H^{(k)}$ are defined in terms of $r_D^{-1}, y^{(k)}$ and $q^{(k)}$ in (40) and (41). Importantly, both $G^{(k)}$ and $H^{(k)}$ are proportional to r_D^{-1} . This implies that the integrals in (52) and (53) are in turn proportional to r_D^{-1} .

We can choose initial settings $y^{(0)}(t)$ and $q^{(0)}(t)$ for the iteration such that $\|y^{(0)}\| \leq R$ and $\|q^{(0)}\| \leq R$, where $R = \max\{R_1 + C_0, R_2\}$ with $R_1 = \|y^{(0)} - e^{(A-VP)(t-t_0)} y_0\|$, $C_0 = \|e^{(A-VP)(t-t_0)} y_0\|$ and



Fig. 1. Closed-loop control system.

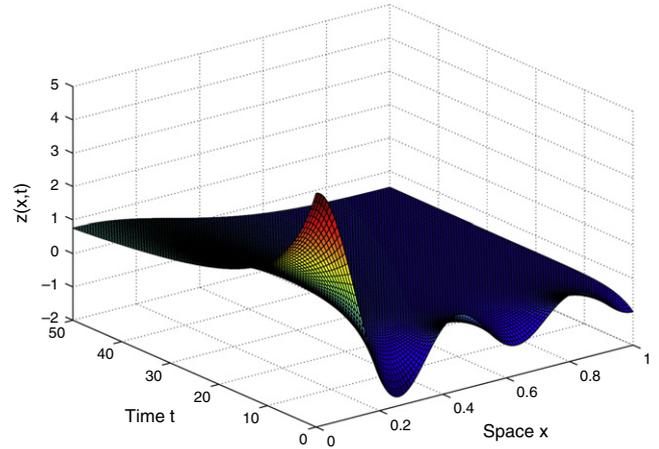


Fig. 2. Uncontrolled dynamics of system (3).

$R_2 = \|q^{(0)}\|$. Based on (40) and (41), we have

$$\|G^{(0)}\| \leq \frac{C_1}{r_D} R^3, \quad \|H^{(0)}\| \leq \frac{C_2}{r_D} R^3, \quad (54)$$

where the constants C_1 and C_2 are given by $C_1 = \|K\|^2(\|P\| + 1)$, $C_2 = \|K\|(\|P\| + 1)(\|K\| + \|K^T P\|)$. Then, we can carry out the first iteration based on (52) and (53) to obtain $y^{(1)}$ and $q^{(1)}$, which are bounded as

$$\|q^{(1)}(t)\| \leq \frac{C_3}{r_D} R^3, \quad \|y^{(1)}(t) - e^{(A-VP)(t-t_0)} y(t_0)\| \leq \frac{C_4}{r_D} R^3,$$

where

$$C_3 = \max_{t \in [t_0, t_f]} \int_t^{t_f} \|e^{(A-VP)^T(t-\tau)}\| d\tau (\|P\| C_1 + C_2),$$

$$C_4 = \max_{t \in [t_0, t_f]} \int_{t_0}^t \|e^{(A-VP)(t-\tau)}\| d\tau (\|V\| C_3 + C_1).$$

Thus, if we choose $r_D \geq \max\{C_1 R^2, C_2 R^2, C_3 \frac{R^3}{R_2}, C_4 \frac{R^3}{R_1}\}$, we obtain $\|y^{(1)}(t) - e^{(A-VP)(t-t_0)} y(t_0)\| \leq R_1$ and $\|q^{(1)}(t)\| \leq R_2$ ($\Rightarrow \|y^{(1)}\| \leq R$ and $\|q^{(1)}\| \leq R$). Now that $\|y^{(1)}\| \leq R$ and $\|q^{(1)}\| \leq R$, the same argument can be repeated with this choice of r_D to prove that $\|G^{(1)}\| \leq R$, $\|H^{(1)}\| \leq R$, $\|y^{(2)}\| \leq R$ and $\|q^{(2)}\| \leq R$.

Generalizing, we have proved that for any $k \geq 0$

$$\|G^{(k)}\| \leq R, \quad \|H^{(k)}\| \leq R, \quad \|q^{(k+1)}\| \leq R, \quad \|y^{(k+1)}\| \leq R,$$

as long as $r_D \geq \max\{C_1 R^2, C_2 R^2, C_3 \frac{R^3}{R_2}, C_4 \frac{R^3}{R_1}\}$. Therefore, the sequences $y^{(k)}$ and $q^{(k)}$ are uniformly bounded in k , which implies that the elements of $T^{(k)}$, and $\max |\sigma(T^{(k)})|$, are uniformly bounded in k . \square

6. Simulation study

Closing the control loop with the iteration based feedback laws is not as direct as in the finite dimensional case (see Fig. 1). After the

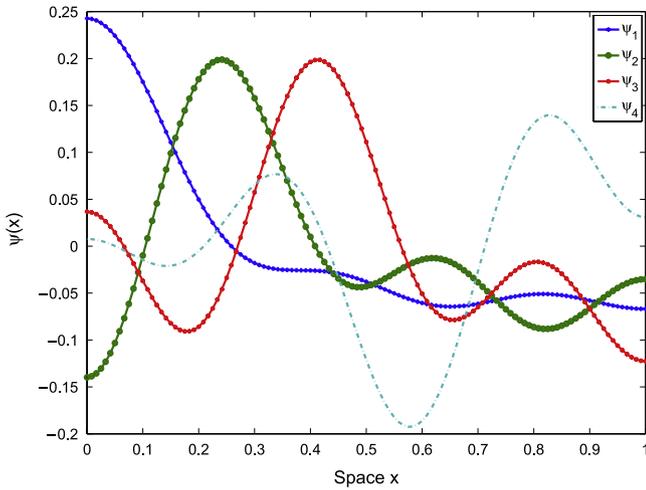


Fig. 3. The first four ($l = 4$) dominant POD modes.

N th iteration, we can obtain the suboptimal feedback controllers

$$u_l^{(N)} = -r_l^{-1} F^T [P y + q^{(N)}], \tag{55}$$

$$u_D^{(N)} = -r_D^{-1} (K y)^T [P y + q^{(N)}], \tag{56}$$

based on (30) and (31), where $y(t)$ is the finite dimensional approximation, with respect to the l POD modes, of $z(x, t)$. Before being able to implement the feedback laws in the original system (3), with the physical domain defined over $0 \leq x \leq L = 1$, we need to rewrite the control laws in terms of $z(x, t)$.

In order to carry out the simulation of the closed-loop system composed by the nonlinear PDE system and the proposed controllers, we use a higher-order approximation $\mathbf{Y}(t)$ of $z(x, t)$ based on the pseudo-spectral method. The state evolution can be expanded by a series of normalized harmonic functions, $z(x, t) \approx \sum_{j=1}^e \beta_j(t) \phi_j(x)$, where $\phi_j(x) = \frac{\cos(j\pi x)}{\sqrt{2}}$, which satisfy the homogeneous Neumann boundary conditions. Then, we can derive a higher-order finite dimensional system using the Galerkin projection method

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y} + \mathbf{K}u_D^{(N)} + \mathbf{F}u_l^{(N)}, \tag{57}$$

where the system vector is defined by $\mathbf{Y} = [\beta_1, \dots, \beta_e]^T$, with $e \gg l$. The system matrices can be obtained by following the same Galerkin projection method discussed in Section 4 but replacing the POD modes by harmonic basis functions, i.e.,

$$[\mathbf{A}]_{ij} = - \int_0^1 \left(\zeta \frac{\partial(\gamma \phi_j)}{\partial x} \frac{\partial \phi_k}{\partial x} + \lambda \phi_j(x) \phi_k(x) \right) dx,$$

$$[\mathbf{K}]_{ij} = - \int_0^1 \zeta \frac{\partial(\gamma \phi_j)}{\partial x} \frac{\partial \phi_k}{\partial x}, \quad [\mathbf{F}]_j = \int_0^1 \xi(x) \phi_j(x) dx,$$

$$[\mathbf{Y}_0]_j = \int_0^1 z(x, t_0) \phi_j(x) dx.$$

By noting that $\alpha_i = \int_0^1 z(x, t) \psi_i(x) dx$ which can be rewritten as $\alpha_i = \sum_{j=1}^e \beta_j \int_0^1 \phi_j(x) \psi_i(x) dx$ and introducing $\mathbf{C} \in \mathbb{R}^{l \times e}$, where the elements are defined by $[\mathbf{C}]_{ij} = \int_0^1 \phi_j(x) \psi_i(x) dx$, then we have $y = \mathbf{C}\mathbf{Y}$. Thus, we can formulate the feedback laws in terms of the new state vector \mathbf{Y} ,

$$u_l = -r_l^{-1} F^T (\mathbf{P}\mathbf{C}\mathbf{Y} + q^*), \tag{58}$$

$$u_D = -r_D^{-1} (\mathbf{K}\mathbf{C}\mathbf{Y})^T (\mathbf{P}\mathbf{C}\mathbf{Y} + q^*). \tag{59}$$

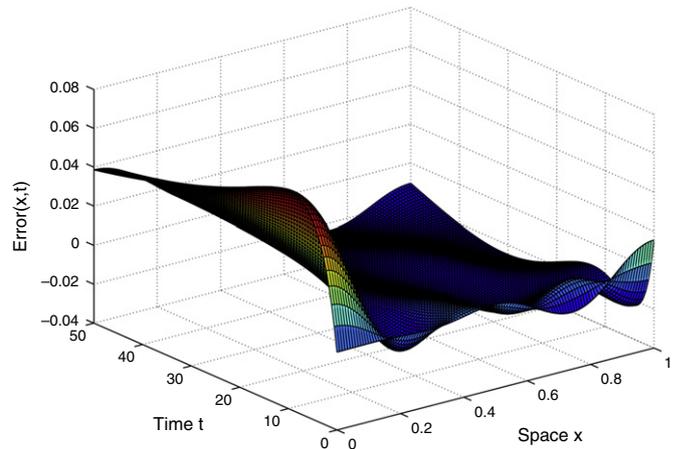


Fig. 4. Error between PDE and reduced order ODE systems.

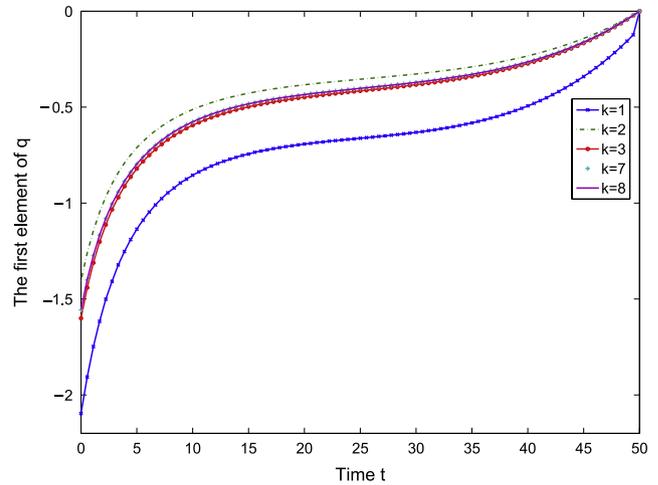


Fig. 5. Convergence of the first element of q .

Therefore, the closed-loop system (Fig. 1) becomes

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y} - r_D^{-1} \mathbf{K}\mathbf{Y}(\mathbf{K}\mathbf{C}\mathbf{Y})^T (\mathbf{P}\mathbf{C}\mathbf{Y} + q^*) - r_l^{-1} \mathbf{F}\mathbf{F}^T (\mathbf{P}\mathbf{C}\mathbf{Y} + q^*).$$

First we simulate the system (3) over $t_0 = 0 \leq t \leq t_f = 50$ with the following parameters: $\gamma(x) = 1$, $\zeta(x) = 10^{-3}$, $\xi(x) = \cos(\pi x)$, $\lambda(x) \equiv 0$, $\varphi(x) = \sum_{k=1}^5 \cos(k\pi x)$, $u_l(t) = u_D(t) = 0$. The simulation is carried out using $e = 36$ harmonic basis functions in the pseudo-spectral approximation. The system evolution and the dominant POD modes (reconstructed via splines interpolation) are shown in Figs. 2 and 3, respectively. By using the first four POD modes ($l = 4$) we can construct the low dimensional bilinear system that is used to synthesize the suboptimal controller. The approximation error due to the model reduction is shown in Fig. 4. For the design of the suboptimal controller based on the iterative scheme proposed in this paper we choose $r_l = 1$, $r_D = 100$, $S = 40\mathbf{I}$ and $Q = 0.15\mathbf{I}$, where \mathbf{I} represents an identity matrices of dimension l . Fig. 5 shows the first element of the feed-forward term $q(t)$ as a function of the iteration number. After 8 iteration the scheme converges and the obtained feedback laws (58) and (59) are implemented. The simulation of the evolution of the closed-loop PDE system is shown in Fig. 6. The controller action shown in Fig. 7 can effectively enhance the dissipation of the evolutionary system. A comparison between controlled and uncontrolled cases at $t = t_f = 50$ is shown in Fig. 8.

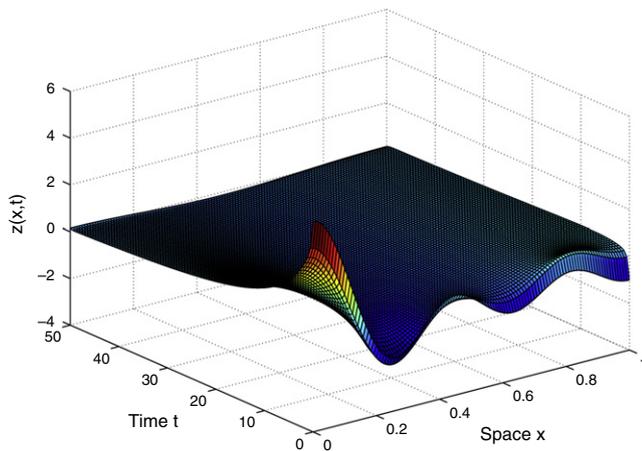


Fig. 6. Closed-loop PDE system simulation.

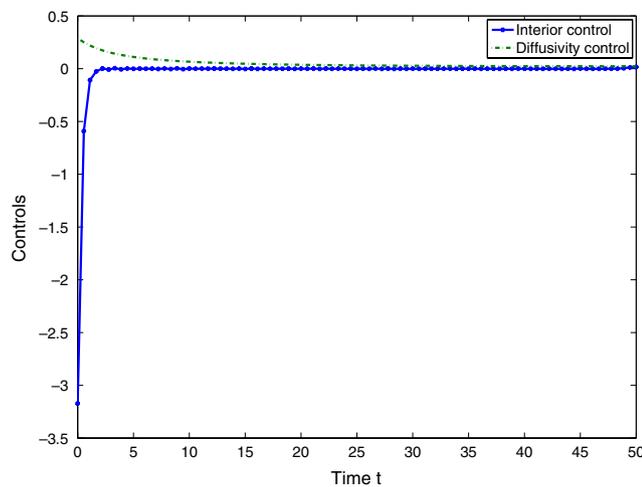


Fig. 7. Interior and diffusivity control.

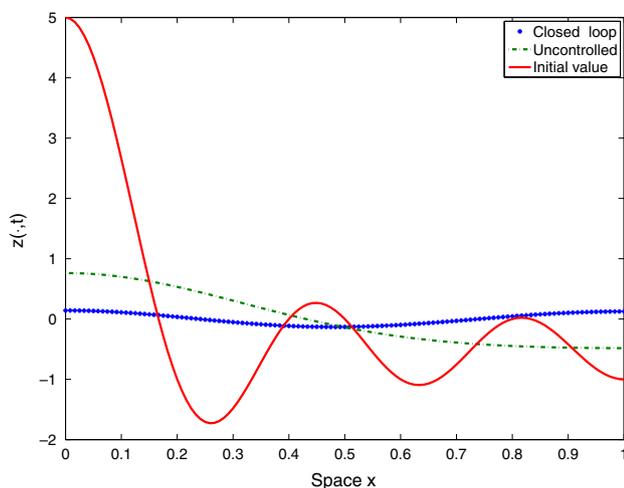


Fig. 8. Comparison of final profiles.

7. Conclusions

In this paper we study a finite-time optimal control problem of a parabolic system arising in plasma current transport in tokamak

plasmas. Two types of actuation mechanisms are considered: diffusivity and interior controls. By using the POD technique, we derive a low dimensional dynamical system which governs the dominant dynamics of the original parabolic system. The reduced order system is of a bilinear form. We propose a convergent successive scheme based on the Picard approximation to compute the solution of a finite-time suboptimal control problem defined for the reduced order bilinear system. This algorithm avoids repeated numerical computation of the Riccati equation at each iteration step by introducing an iteration scheme for the feed-forward control component. In terms of the number of ODEs required to solve the Riccati matrix equation (P -equation) and the feed-forward vector equation (q -equation), this method can decrease the number of ODEs to be computed at each iteration step from l^2 to l . Simulation studies using the original PDE system show the effectiveness of the model reduction technique and the successive suboptimal control scheme.

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Chao Xu is an Assistant Professor in the Department of Control Science & Engineering and the Institute of Cyber-Systems & Control at Zhejiang University. He received his B.Sc. degree (Mechanical Engineering) from Shenyang Aerospace University in 2002 and his M.Sc. degree (Mathematics) from Zhejiang University in 2005. He obtained his Ph.D. degree (Mechanical Engineering) from Lehigh University in 2010. His research interests include control of distributed parameter systems, model order reduction for large-scale systems and plasma control in magnetic fusion.



Yongsheng Ou is a Professor at Shenzhen Institute of Advanced Technology, Chinese Academy of Sciences. He holds a B.Sc. degree in Mechanical & Electrical Engineering (Beijing University of Aeronautics and Astronautics, 1995) and a M.Sc. degree in Electrical Engineering (Institute of Automation, Chinese Academy of Sciences, 1998). Ou received a Ph.D. degree in Automation & Computer-Aided Engineering from the Chinese University of Hong Kong in 2004 and a Ph.D. degree in Mechanical Engineering & Mechanics from Lehigh University in 2010. He is a coauthor of the monograph on Control of Single Wheel Robots (Springer, 2005). His research interests include control of distributed parameter systems with applications to fusion reactors, learning control by demonstration, and computer vision.



Eugenio Schuster is Associate Professor of Mechanical Engineering and Mechanics at Lehigh University. He holds undergraduate degrees in Electronic Engineering (Buenos Aires University, 1993) and Nuclear Engineering (Balseiro Institute, 1998). He obtained his M.Sc. (2000) and Ph.D. (2004) degrees in Mechanical & Aerospace Engineering from University of California San Diego. Schuster is the recipient of the NSF Career Award. His research interests are in distributed parameter and nonlinear control systems, with applications that include fusion reactors, plasmas, and magnetohydrodynamic flows.