Observer-based Stabilization of an Unstable Parabolic PDE Using the Pseudospectral Method and Sturm-Liouville Theory

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***Abstract*— The stabilization of an unstable linear parabolic partial differential equation (PDE) system with both Neumann boundary control and interior control is considered in this work. Point output measurement is available at one end of the physical domain. The choice of a proportional output feedback boundary control is justified by Lyapunov analysis while the design of the interior control is carried out based on the Sturm-Liouville theory. A proportional state feedback is proposed for the interior control with a symmetric kernel function, and the pseudospectral method is used to solve the stability conditions governed by the Sturm-Liouville systems. In addition, an observer is designed using the point measurement at one end of the physical domain, and used to propose an observer–based feedback controller for the PDE system. Both controller and observer gains are designed numerically to make the eigenvalues of the associated Sturm-Liouville problems stable. Simulations show the effectiveness of the proposed controller.**

I. INTRODUCTION

The stabilization of linear distributed parameter systems has been an active area of research for more than one decade now (e.g., [1], [2], [3], [4], [5] and references therein). In this paper, simultaneous boundary and interior control are employed for the the stabilization of an unstable linear parabolic partial differential equation (PDE) system described by

$$\begin{cases} \psi_t = \psi_{xx} + f(x)v(t) + \lambda\psi, \\ \psi_x(0,t) = 0,\ \psi_x(1,t) = w(t), \end{cases} \tag{1}$$

$$y(t) = \psi(1,t), \tag{2}$$

over the spatial–temporal domain $Q_T = \{(x,t) : x \in \Omega = [0,1], t \in [0,\infty)\}$; where $\psi(x,t)$ is the system state, $y(t)$ is the system output, $\lambda(> 0)$ is a sufficiently large number to make the system unstable, $f(x)$ is the input function with respect to the interior control $v(t)$, and $w(t)$ is the Neumann boundary control. After justifying the choice of a proportional output feedback boundary control by Lyapunov analysis, the design of the interior control is carried out based on the Sturm-Liouville theory.

Model reduction is commonly implemented via the method of lines (MOL) ([6], [7], [8], [9], [10]) in order to obtain finite dimensional lumped–parameter representations of the original PDE systems. This is referred to as the reduce-then-design approach in [11]. The control of the reduced-order system can be carried out by applying different existing finite dimensional control techniques. However, a drawback of the reduce-then-design method is the inherent loss of information due to the truncation before control design. Moreover, the order of the model truncation is a trade-off between model accuracy and real time computation. A linear approach based on the reduced order model is carried out in [4] to study the problem of global stabilization of a semi-linear dissipative evolution equation by the Lyapunov technique in finite dimensional controls. Bounds of the Lyapunov energy with respect to the neglected higher order components are obtained to avoid the spillover phenomenon due to nonlinear couplings. Model reduction using the approximate inertial manifold of dissipative systems [5] ensures global stabilization by considering the higher-mode-components as a singular perturbation to the finite dimensional lower-mode-components.

Instead of implementing model reduction before control design, we follow a reduce-then-design method [11] in this work. We consider a proportional type interior control for the unstable PDE system. An integral operation for the product of the proportional feedback kernel gain and the system state is used for the PDE stabilization in this paper, e.g., $v(t) = \int_0^1 k_v f(y)\psi(y,t)dy$, where $f(y)$ is the control actuation function and k_v is the to-be-designed gain. By substituting the proposed proportional control law into the unstable PDE system, we use the variable separation method to obtain a self-adjoint Sturm-Liouville problem associated with the closed–loop system (i.e., the spectral conditions of the closed-loop C_0-semigroup), which includes the to-be-designed feedback controller gain (e.g., kernel function). The closed–loop Sturm-Liouville system is an integro–differential-type two boundary value problem which does not admit an analytical solution in general, and numerical methods are necessary for its solution. Using the eigenfunctions obtained from the uncontrolled Sturm-Liouville problem (relevant to the boundary feedback design), we apply the pseudospectral method to rewrite the controlled Sturm-Liouville problem as a finite dimensional matrix eigenvalue problem, which can be equivalently considered as a pole placement problem for PDE systems.

The spatial–temporal state information needed in the proposed proportional control law makes it impractical since this information is usually not available. Therefore, an observer to estimate the spatial–temporal state information is designed exploiting the availability of point measurement at one end of the physical domain. Point measurement by locating sensors at specific points of interests in the physical domain is common and feasible in engineering practice. The estimation error dynamics define a non-self-adjoint Sturm-Liouville

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problem, which includes the to-be-designed observer gain. Similarly to the controller case, Galerkin projection is used to reduce the Sturm-Liouville problem to a pole placement problem. In this case, the eigenfunctions obtained by solving the Sturm-Liouville problem associated with the uncontrolled error dynamics are used during the Galerkin projection.

Sano employed output feedback in [12] to stabilize the first order heat exchanger PDEs using Huang's result on the spectrum determined growth assumption. More work along this line can be found in [13], [14] and references therein. However, the analytical study of the spectra associated with the closed-loop C_0-semigroup is complicated. The second-order nature of the parabolic PDE under consideration in this work makes the spectral analysis even more complex when designing the state observer based on the boundary measurement (by duality the feedback mechanism is similar to that in [12]). The contribution of our work resides on the development of numerical algorithms for the design of an explicit control law with a proportional feedback kernel function (infinite dimensional proportional control) which stabilize the infinite dimensional system. The observer design can be seen as the dual formulation of the stabilization problem. Both the controller and observer designs are formulated as Sturm-Liouville problems that can be solved with the pseudospectral-Galerkin method.

The paper is organized as follows. We present the boundary control in Section II. An infinite-dimensional interior control is presented in Section III. A simulation study for the infinite-dimensional controller is carried out in Section IV, where both the numerical scheme and a numerical example are discussed. We close this paper by stating conclusions and future research topics in Section V.

II. Boundary Control

We consider the control Lyapunov function

$$V(\psi) := \frac{1}{2}\|\psi\|^2 = \frac{1}{2}\int_0^1 |\psi(x,t)|^2 dx := \frac{1}{2}\langle \psi, \psi\rangle, \tag{3}$$

where $\|\cdot\|$ is the usual norm in $L^2(0,1)$, and $\langle\cdot,\cdot\rangle$ is the inner product. We compute the time derivative of the control Lyapunov function V to obtain

$$\begin{aligned}\dot{V} &= \int_\Omega \psi\psi_t = \int_\Omega \psi\left[\psi_{xx} + f(x)v(t) + \lambda\psi\right] \\ &= \psi(1)\psi_x(1) - \int_\Omega \psi_x^2 dx + \int_\Omega \left[\lambda\psi^2 + v(t)f\psi\right] dx.\end{aligned} \tag{4}$$

We can find that $\dot{V}$ can be positive to make the system (1) unstable when λ is sufficient large. To enhance the negativeness of $\dot{V}$, we can let $w(t) = -k_w\psi(1,t)$, where k_w is the feedback gain. Although it may be possible to stabilize the unstable system (1) without using interior control by carefully choosing k_w high enough, in this paper we set $k_w = 1$ to avoid high boundary control action and follow a combined boundary-interior control approach, i.e.,

$$w(t) = -\psi(1,t). \tag{5}$$

Substituting the feedback law (5) into (1), the PDE system becomes

$$\begin{cases}\psi_t = \psi_{xx} + \lambda\psi + fv, \\ \psi_x(0,t) = \psi_x(1,t) + \psi(1,t) = 0.\end{cases} \tag{6}$$

This is an unstable system if λ is sufficiently large, and we will use the interior control v to stabilize it in this work.

III. Infinite-dimensional Interior Feedback

A. Control Design

We propose an interior feedback control with the following proportional kernel form:

$$v(t) = -\int_\Omega k_v f(y)\psi(y,t)dy, \tag{7}$$

where the feedback gain k_v is to be determined. Then, the closed–loop system takes the form of

$$\begin{cases}\psi_t = \psi_{xx} - \int_\Omega k_v f(x)f(y)\psi(y,t)dy + \lambda\psi, \\ \psi_x(0) = \psi_x(1) + \psi(1) = 0.\end{cases} \tag{8}$$

Theorem 1: Given the unstable system (1) and the boundary feedback law (5), the interior feedback (7) can stabilize the system if the eigenvalues of the following system satisfies $\mu < 0$:

$$\begin{cases}X'' - \int_\Omega k_v f(x)f(y)X(y)dy + \lambda X = \mu X, \\ X'(0) = X'(1) + X(1) = 0.\end{cases} \tag{9}$$

Proof: Using the variable separation method $(\psi(x,t) = X(x)T(t))$, we can rewrite the system (8) as

$$\frac{\dot{T}(t)}{T(t)} = \frac{\left[X''(x) - \int_\Omega k_v f(x)f(y)X(y)dy + \lambda X(x)\right]}{X(x)} = \mu,$$

with the boundary condition given by $X'(0)T(t) = [X'(1) + X(1)]\,T(t) = 0$. Thus, we obtain the eigenvalue problem (8) and the temporal equation $\dot{T}(t) - \mu T(t) = 0$ which has exponentially stable solution if $\mu < 0$. ■

Therefore, the stabilization problem becomes to solve the integro-differential equation (9). Based on the feedback kernel function chosen in (7), we can prove that all the eigenvalues governed by (8) are real numbers.

Theorem 2: The eigenvalues of the Sturm-Liouville system (9) are real numbers.

Proof: We introduce the operator S_1 associated with (9) $(S_1 g)(x) = \frac{d^2 g}{dx^2} - \int_\Omega k_v f(x)f(y)g(y)dy + \lambda g(x)$, with the domain $D(S_1) = \{g \in H^2; g'(0) = g'(1) + g(1) = 0\}$ and $H^2 = \{g; g, g' \text{ and } g'' \in L^2(0,1)\}$. We can show that S_1 is self-adjoint, i.e, given $g_1, g_2 \in D(S_1)$, it satisfies

$\langle g_2, S_1 g_1\rangle = \langle g_1, S_1 g_2\rangle$:

$$\begin{aligned}\langle g_2, S_1 g_1\rangle &= \int_0^1 \frac{d^2 g_1}{dx^2} g_2 dx \\ &\quad - \int_0^1\int_0^1 k_v f(x) f(y) g_1(y) dy g_2(x) dx + \int_0^1 \lambda g_1 g_2 dx \\ &= -g_1(1) g_2(1) - \int_0^1 g_1' g_2' dx \\ &\quad - k_v \int_0^1 f(y) g_1(y) dy \int_0^1 f(x) g_2(x) dx + \int_0^1 \lambda g_1 g_2 dx \\ &= \int_0^1 g_1 g_2'' dx \\ &\quad - k_v \int_0^1 f(x) g_1(x) dx \int_0^1 f(y) g_2(y) dy + \int_0^1 \lambda g_2 g_1 dx \\ &= \langle g_1, S_1 g_2\rangle.\end{aligned} \tag{10}$$

It is known that self-adjoint operators have real eigenvalues. ■

Therefore, the stabilization problem is to find a feedback gain k_v such that all the eigenvalues of the operator S_1 are negative. However, the associated Sturm-Liouville problem for feedback design can not be solved explicitly for a general control function $f(x)$ and numerical methods are necessary. The Sturm–Liouville problem of (8), when $k_v = 0$, is

$$\phi_n'' = -\gamma_n^2 \phi_n,\ \phi_n'(0) = \phi_n'(1) + \phi_n(1) = 0,\ n \in \mathbb{N}. \tag{11}$$

We assume that the solution of (9) can be approximated as

$$X(x) \approx \sum_{i=1}^{I_c} a_i \phi_i(x), \tag{12}$$

where I_c is a truncation number of the infinitely many basis functions provided by (11), and $a_i, (i = 1, 2, \cdots, I_c)$ are constants. Then, we can multiply both sides of (9) by ϕ_j and integrate over $[0, 1]$ to obtain

$$\begin{aligned}& - \sum_{i=1}^{I_c} a_i \phi_i(1)\phi_j(1) - \int_0^1 \sum_{i=1}^{I_c} a_i \phi_i'(x)\phi_j'(x) dx \\ &\quad - k_v \sum_{i=1}^{I_c} a_i \int_0^1\int_0^1 f(x) f(y) \phi_i(y)\phi_j(x) dy dx \\ &\quad + \lambda \sum_{i=1}^{I_c} a_i \int_0^1 \phi_i(x)\phi_j(x) dx \\ &= - \sum_{i=1}^{I_c} [\phi_i(1)\phi_j(1)] a_i - \sum_{i=1}^{I_c} \left[\int_0^1 \phi_i'(x)\phi_j'(x) dx\right] a_i \\ &\quad - k_v \sum_{i=1}^{I_c} [f_i f_j] a_i + \lambda \sum_{i=1}^{I_c} \left[\int_0^1 \phi_i(x)\phi_j(x) dx\right] a_i \\ &= \mu \sum_{i=1}^{I_c} \left[\int_0^1 \phi_i(x)\phi_j(x) dx\right] a_i.\end{aligned} \tag{13}$$

We introduce the matrix notation

$$\mathbf{A}_1(i,j) = \phi_i(1)\phi_j(1), \quad \mathbf{A}_2(i,j) = \int_0^1 \phi_i'(x)\phi_j'(x) dx \tag{14}$$

$$\mathbf{A}_3(i,j) = f_i f_j = \int_0^1\int_0^1 f(x) f(y)\phi_i(y)\phi_j(x) dy dx, \tag{15}$$

$$\mathbf{A}_4(i,j) = \int_0^1 \phi_i(x)\phi_j(x) dx, \quad \mathbf{a} = [a_1, \cdots, a_{I_c}]^T, \tag{16}$$

and rewrite (13) to obtain the finite dimensional representation of (9):

$$(-\mathbf{A}_1 - \mathbf{A}_2 - k_v \mathbf{A}_3 + \lambda \mathbf{A}_4)\,\mathbf{a} = \mu \mathbf{A}_4 \mathbf{a}. \tag{17}$$

To make equation (17) have non-trivial solutions, we must satisfy the following equation with respect to μ:

$$\det((\mu - \lambda)\mathbf{A}_4 + \mathbf{A}_1 + \mathbf{A}_2 + k_v \mathbf{A}_3) = 0. \tag{18}$$

Therefore, the stabilization problem is to find a control gain k_v to place the roots of (18) on the left half plane ($\Re(\mu) < 0$).

B. Observer Design

We can assume that the observer takes the following form

$$\begin{cases} \widehat{\psi}_t = \widehat{\psi}_{xx} + f(x) v + \lambda \widehat{\psi} + k_o(x)\left[\psi(1) - \widehat{\psi}(1)\right] \\ \widehat{\psi}_x(0) = \widehat{\psi}_x(1) + k_w \psi(1) = 0 \end{cases} \tag{19}$$

where $k_o(x)$ is the observer gain to be designed. We define $e = \psi - \widehat{\psi}$, which is governed by

$$\begin{cases} e_t = e_{xx} + \lambda e - k_o(x) e(1), \\ e_x(0) = e_x(1) = 0. \end{cases} \tag{20}$$

We use the variable separation method $(e(x,t) = Y(x)T(t))$ to obtain the Sturm–Liouville problem of (20),

$$\begin{cases} Y'' - k_o(x) Y(1) + \lambda Y = \nu Y, \\ Y'(0) = Y'(1) = 0, \end{cases} \tag{21}$$

where ν can be a complex number since the operator $S_2\varphi = \frac{d^2\varphi(x)}{dx^2} - k_o(x)\varphi(1) + \lambda\varphi(x)$ over the domain $D(S_2) = \{\varphi \in H^2; \varphi'(0) = \varphi'(1) = 0\}$ is non self-adjoint, i.e., $\langle \varphi_2, S_2\varphi_1\rangle \neq \langle \varphi_1, S_2\varphi_2\rangle$, $\forall \varphi_1, \varphi_2 \in D(S_2)$. We can explicitly solve the eigenvalue problem $S_2\varphi = \nu\varphi$, $\nu \in \mathbb{C}$, when $k_o(x)$ is a simple function, such as a constant or a harmonic function as discussed below. However, a numerical approach is needed in the general case.

1) Constant gain: If the feedback gain in the observer is a constant k_o, the Sturm-Liouville problem (21) becomes

$$\frac{d^2\varphi(x)}{dx^2} - k_o\varphi(1) + \lambda\varphi(x) = \nu\varphi(x), \tag{22}$$

$$\frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(1) = 0. \tag{23}$$

We first check that $\nu = \lambda$ is not the eigenvalue. If $\nu = \lambda$, we have the solution $\varphi(x) = C_1 x^2 + C_2 x + C_3$, where C_1, C_2, C_3 are constants determined by $\varphi'(0) = C_2 = 0$, $\varphi'(1) = 2C_1 = 0$, but $\varphi''(x) = 2C_1 = 0 \neq k_o\varphi(1)$. Then $\nu \neq \lambda$ and the general solution is given by

$$\varphi(x) = C_1\cos(\sqrt{\nu - \lambda}x) + C_2\sin(\sqrt{\nu - \lambda}x) + \frac{k_o\varphi(1)}{\nu - \lambda}, \tag{24}$$

where C_1 and C_2 are constants. When $C_1 = C_2 = 0$, we have a constant solution,

$$\varphi(x) = \frac{-k_o\varphi(1)}{\nu - \lambda} \Longrightarrow \nu = -k_o + \lambda. \tag{25}$$

Then, we can find that a constant gain k_o can change the eigenvalue λ. The general solution (24) satisfies the boundary conditions $\varphi'(0) = \varphi'(1) = 0$ where $C_1(\neq 0)$ and $C_2(= 0)$ are constants to be determined by the boundary conditions (23), i.e.,

$$\begin{cases} C_1\sqrt{\nu-\lambda}\sin(0) - C_2\sqrt{\nu-\lambda}\cos(0) = 0, \\ C_1\sqrt{\nu-\lambda}\sin(\sqrt{\nu-\lambda}) - C_2\sqrt{\nu-\lambda}\cos(\sqrt{\nu-\lambda}) = 0. \end{cases} \tag{26}$$

Then, the eigenvalue ν is determined by $\sin(\sqrt{\nu-\lambda}) = 0$ which is independent of k_o. This is an interesting result that shows that a constant gain k_o can only change the eigenvalue corresponding to the first constant eigenfunction. Therefore, to design an effective observer based on the point measurement output, a gain function including positive frequency harmonics is required.

2) Harmonic function gain: We choose sine functions as the feedback gain, i.e., $k_o(x) = \sin(n\pi x)$, $n \in \mathbb{N}$, then (21) becomes

$$\frac{d^2\varphi(x)}{dx^2} - k_o \sin(n\pi x)\varphi(1) + \lambda\varphi(x) = \nu\varphi(x), \tag{27}$$

$$\frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(1) = 0, \tag{28}$$

whose solution is given by

$$\begin{aligned} \frac{\varphi(x)}{k_o\varphi(1)} &= \frac{n\pi\sin\sqrt{\nu-\lambda}x}{\sqrt{\nu-\lambda}(\nu-\lambda-n^2\pi^2)} - \frac{\sin(n\pi x)}{\nu-\lambda-n^2\pi^2} \\ &+ \frac{n\pi\cos(\sqrt{\nu-\lambda}x)(\cos\sqrt{\nu-\lambda}-\cos(n\pi))}{\sqrt{\nu-\lambda}(\nu-\lambda-n^2\pi^2)\sin(\sqrt{\nu-\lambda})}, \end{aligned} \tag{29}$$

when $\nu \neq \lambda$, $\nu - \lambda \neq n^2\pi^2$ and $\nu - \lambda \neq n\pi$. The other cases are

$$\nu = \lambda: \quad \frac{\varphi(x)}{k_o\varphi(1)} = \frac{\sin(n\pi x) - 1}{n^2\pi^2}, \tag{30}$$

$$\begin{aligned} \nu = \lambda + n^2\pi^2: \quad \frac{\varphi(x)}{k_o\varphi(1)} &= \frac{e^{n\pi(1-x)}[e^{n\pi} - (-1)^n]}{2n^2\pi^2(-1+e^{2n\pi})} \\ &- \frac{e^{n\pi x}\left[(-1)^n e^{n\pi} - 1\right]}{2n^2\pi^2(-1+e^{2n\pi})} + \frac{\sin(n\pi x)}{2n^2\pi^2}, \end{aligned} \tag{31}$$

$$\begin{aligned} \nu = \lambda + n\pi: \quad \frac{\varphi(x)}{k_o\varphi(1)} &= \frac{e^{\sqrt{n\pi}x}\left[-1+(-1)^n e^{\sqrt{n\pi}}\right]}{(1+n\pi)\sqrt{n\pi}\left(-1+e^{2\sqrt{n\pi}}\right)} \\ &- \frac{e^{\sqrt{n\pi}(1-x)}\left[(-1)^n - e^{\sqrt{n\pi}}\right]}{(1+n\pi)\sqrt{n\pi}\left(-1+e^{2\sqrt{n\pi}}\right)} + \frac{\sin(n\pi x)}{n\pi(1+n\pi)}. \end{aligned} \tag{32}$$

By making $x = 1$ in (29), we can obtain the characteristic equation for ν ($n \in \mathbb{N}$, $\nu \neq \lambda$, $\nu - \lambda \neq n^2\pi^2$ and $\nu - \lambda \neq n\pi$):

$$\sqrt{\nu-\lambda}(\nu-\lambda-n^2\pi^2)\sin\sqrt{\nu-\lambda} + k_o n\pi\left[(-1)^n\cos\sqrt{\nu-\lambda} - 1\right] = 0. \tag{33}$$

We neglect the other three cases in (30)–(32), since the eigenvalue ν is positive and not of our interest. Therefore, the eigenvalues of the operator S_2 is given by $\sigma(S_2) := \{\nu : \text{ equation (33)}\} \cap \{\nu : \nu \neq \lambda\} \cap \{\nu : \nu - \lambda \neq n\pi\} \cap \{\nu : \nu - \lambda \neq n^2\pi^2\}$. The stabilization problem is to find a feedback gain k_o such that the roots of (33) in $\sigma(S_2)$ reside on the left half plane. However, this is a transcendental complex equation and not always able to be solved explicitly. Therefore, numerical computation is necessary to solve the Sturm-Liouville problem (21) associated with the observer design problem.

3) General function gain - Numerical approach: The Sturm–Liouville problem of (20), when $k_o(x) = 0$, is

$$\varphi_n'' = -\omega_n^2\varphi_n, \; \varphi_n'(0) = \varphi_n'(1) = 0, \; n \in \mathbb{N}. \tag{34}$$

We assume the solution of (21) can be expressed as

$$Y(x) \approx \sum_{i=1}^{I_o} b_i\varphi_i(x), \tag{35}$$

where I_o is a truncation number of the infinitely many basis functions provided by (34), and $b_i, (i = 1, 2, \cdots, I_o)$ are constants. Then, we can multiply both sides of (21) by φ_j and integrate over $[0, 1]$ to obtain

$$\begin{aligned} -\int_0^1 \sum_{i=1}^{I_o} b_i\varphi_i'(x)\varphi_j'(x)dx - \sum_{i=1}^{I_o} b_i\varphi_i(1)\int_0^1 k_o(x)\varphi_j(x)dx \\ = (\nu-\lambda)\int_0^1 \sum_{i=1}^{I_o} b_i\varphi_i(x)\varphi_j(x). \end{aligned} \tag{36}$$

We introduce the matrix notation

$$\mathbf{B}_1(i,j) = \langle\varphi_i, \varphi_j\rangle, \tag{37}$$

$$\mathbf{B}_2(i,j) = \langle\varphi_i', \varphi_j'\rangle, \tag{38}$$

$$\mathbf{B}_3(i,j) = \varphi_i(1)k_{o,j}, \tag{39}$$

where $k_{o,j} = \int_0^1 k_o(x)\varphi_j(x)dx$, and rewrite (36) to obtain the finite dimensional representation of (21):

$$(\lambda\mathbf{B}_1 - \mathbf{B}_2 - \mathbf{B}_3)\,\mathbf{b} = \nu\mathbf{B}_1\mathbf{b}. \tag{40}$$

To ensure that equation (40) has non-trivial solution $\mathbf{b} := [b_1, b_2, \cdots, b_{I_o}]^T$, we require

$$\det((\lambda-\nu)\mathbf{B}_1 - \mathbf{B}_2 - \mathbf{B}_3) = 0. \tag{41}$$

Therefore, the observer design problem is to find $k_{o,j}$ such that the roots of (41) reside on the left half plane, i.e., $\Re(\nu) < 0$.

IV. Simulation Study

A. Numerical Approach

The closed–loop system is governed by the following coupled PDEs:

$$\begin{cases} \psi_t = \psi_{xx} - f(x)\int_0^1 k_v f(y)\psi(y,t)dy + \lambda\psi, \\ \qquad\qquad + f(x)\int_0^1 k_v f(y)e(y,t)dy, \\ e_t = e_{xx} + \lambda e - k_o(x)e(1), \\ \psi_x(0) = \psi_x(1) + \psi(1) = 0, \\ e_x(0) = e_x(1) = 0. \end{cases} \tag{42}$$

Defining an operator

$$S_0 e := f(x)\int_0^1 k_v f(y)e(y,t)dy, \tag{43}$$

we can rewrite the closed–loop system (42) as

$$\frac{d}{dt}\begin{pmatrix}\psi\\ e\end{pmatrix} = \begin{pmatrix} S_1 & S_0\\ 0 & S_2\end{pmatrix}\begin{pmatrix}\psi\\ e\end{pmatrix} \tag{44}$$

where the separation principle holds (see, e.g., [1]) to ensure closed–loop stability.

For the numerical simulation, we can use the Galerkin method to solve the closed–loop system (42). We make the following expansion

$$\psi^{(I_N)} = \sum_i^{I_N} z_i(t)\phi_i(x), \quad e^{(I_N)} = \sum_i^{I_N}\varepsilon_i(t)\varphi_i(x), \tag{45}$$

where the basis functions $\{\phi_i\}_{i=1}^{I_N}$ and $\{\varphi_i\}_{i=1}^{I_N}$ solve (11) and (34), respectively. We note that the index "I_N" in deriving the numerical scheme in this Section is always chosen larger than or equal to "I_c" and "I_o" in Section III. Now we substitute (45) into the system (42) and use Galerkin projection to obtain

$$\begin{cases} \sum_{i=1}^{I_N}\langle\phi_i,\phi_j\rangle\dot{z}_i = -\sum_{i=1}^{I_N}\phi_i(1)\phi_j(1)z_i - \sum_{i=1}^{I_N}\langle\phi_i',\phi_j'\rangle z_i \\ \qquad - k_v\sum_{i=1}^{I_N} f_i f_j z_i + \lambda\sum_{i=1}^{I_N}\langle\phi_i,\phi_j\rangle z_i \\ \qquad + k_v f_j\int_0^1 f(y)e^{(I_N)}(y,t)dy, \\ \sum_{i=1}^{I_N}\langle\varphi_i,\varphi_j\rangle\dot{\varepsilon}_i = -\sum_{i=1}^{I_N}\langle\varphi_i',\varphi_j'\rangle\varepsilon_i + \lambda\sum_{i=1}^{I_N}\langle\varphi_i,\varphi_j\rangle\varepsilon_i \\ \qquad - \sum_{i=1}^{I_N}\varphi_i(1)k_{o,j}\varepsilon_i. \end{cases} \tag{46}$$

We first solve the ε–equations in (46), then substitute $e^{(I_N)}(x,t) = \sum_{i=1}^{I_N}\varepsilon_i(t)\varphi_i(x)$ into the z– equations in (46) to solve the state equations. Defining

$$\mathbf{z} = (z_1, z_2, \cdots, z_{I_N})^T,\ \varepsilon = (\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_{I_N})^T, \tag{47}$$

$$\mathcal{A}_1(i,j) = \phi_i(1)\phi_j(1),\ \mathcal{A}_2(i,j) = \int_0^1 \phi_i'(x)\phi_j'(x)dx \tag{48}$$

$$\mathcal{A}_3(i,j) = f_i f_j = \int_0^1\int_0^1 f(x)f(y)\phi_i(y)\phi_j(x)dydx, \tag{49}$$

$$\mathcal{A}_4(i,j) = \int_0^1 \phi_i(x)\phi_j(x)dx, \tag{50}$$

$$\mathcal{B}_1(i,j) = \langle\varphi_i,\varphi_j\rangle, \quad \mathcal{B}_2(i,j) = \langle\varphi_i',\varphi_j'\rangle, \tag{51}$$

$$\mathcal{B}_3(i,j) = \varphi_i(1)k_{o,j},\ \mathcal{F}(j) = f_j\int_0^1 f(y)e^{(I_N)}(y,t)dy, \tag{52}$$

we can rewrite system (46) as

$$\begin{cases} \mathcal{A}_4\frac{d\mathbf{z}}{dt} = -(\mathcal{A}_1+\mathcal{A}_2+k_v\mathcal{A}_3-\lambda\mathcal{A}_4)\mathbf{z}+\mathcal{F}(\varepsilon), \\ \mathcal{B}_1\frac{d\varepsilon}{dt} = -(\mathcal{B}_2-\lambda\mathcal{B}_1+\mathcal{B}_3)\varepsilon. \end{cases} \tag{53}$$

Remark 1: It is interesting to note that the closed–loop system (53) can be rewritten in the complex domain by using the Laplace transform (assuming the initial values are zeros):

$$\begin{cases} s\mathcal{A}_4\check{\mathbf{z}} = -(\mathcal{A}_1+\mathcal{A}_2+k_v\mathcal{A}_3-\lambda\mathcal{A}_4)\check{\mathbf{z}}+\check{\mathcal{F}}, \\ s\mathcal{B}_1\check{\varepsilon} = -(\mathcal{B}_2-\lambda\mathcal{B}_1+\mathcal{B}_3)\check{\varepsilon}, \end{cases} \tag{54}$$

where $\check{f}$ is defined as the Laplace transform, i.e.,

$$\check{f}(s) = \int_0^\infty e^{st}f(t)dt, \quad s\in\mathbb{C}. \tag{55}$$

Then, we can obtain the characteristic equations for both the state and observer equations

$$|s\mathcal{A}_4 + (\mathcal{A}_1+\mathcal{A}_2+k_v\mathcal{A}_3-\lambda\mathcal{A}_4)| = 0, \tag{56}$$

$$|s\mathcal{B}_1 + \mathcal{B}_2 - \lambda\mathcal{B}_1 + \mathcal{B}_3)| = 0, \tag{57}$$

which become the design conditions obtained in (18) (if $I_c = I_N$) and (41) (if $I_o = I_N$), respectively.

B. Numerical Examples

In this subsection, we assume the input function $f(x) = \cos(0.86x) + f_h\cos(9.53x)$ and $\lambda = 10$, where f_h is a constant. We solve the Sturm–Liouville problem (11) to obtain the first four eigenvalues $\gamma_1 = 0.86$, $\gamma_2 = 3.43$, $\gamma_3 = 6.44$, $\gamma_4 = 9.53$ and associated eigenfunctions

$$\phi_1(x) = \cos(.86x), \quad \phi_2(x) = \cos(3.43x),$$
$$\phi_3(x) = \cos(6.44x), \quad \phi_4(x) = \cos(9.53x).$$

We solve the Sturm-Liouville problem (34) for $n = 0$ and $n\in\mathbb{N}$: $\nu_n = n\pi$, $\varphi_n(x) = \cos(\nu_n x)$.

The stabilization and observer design problems are to find $k_v\in\mathbb{R}$, and $k_o(x)$ such that (18) and (41) have all the roots on the left half plane, i.e., $\Re(\mu) < 0$, $\Re(\nu) < 0$. For the feedback gain design, we choose $I_c = 3$ and compute the matrices defined in (14)–(16). The characteristic equation (18) becomes

$$c_0\mu^3 + c_1\mu^2 + c_2\mu + c_3 = 0, \tag{58}$$

where the coefficients $c_i, (i = 0,1,2,3)$ are given by

$$c_0 = 0.2176, \tag{59}$$
$$c_1 = 0.2763k_v + 5.2039, \tag{60}$$
$$c_2 = 9.1672k_v - 54.9763, \tag{61}$$
$$c_3 = 15.0723k_v - 109.9055. \tag{62}$$

By using the Hurwitz stability criterion, we can obtain the stability condition with respect to the feedback gain k_v:

$$c_i > 0,\ (i = 2,3), \text{ and } c_1c_2 > c_0c_3 \Rightarrow k_v > 5.9256. \tag{63}$$

We choose that the observer gain function takes the form of $k_o(x) = a + b\cos(\pi x)$. Let $I_c = 3$, then we compute the matrices defined in (37)–(39). The characteristic equation (41) becomes

$$d_0\nu^3 + d_1\nu^2 + d_2\nu + d_3 = 0, \tag{64}$$

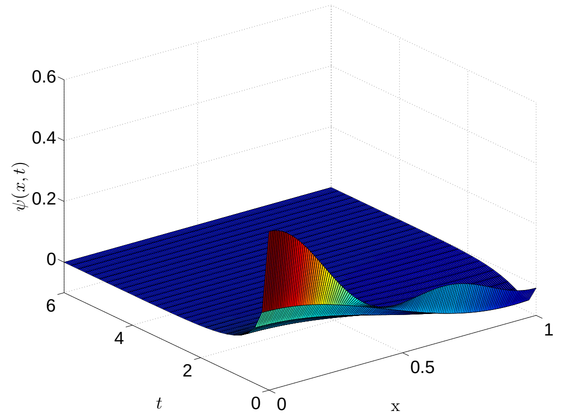

Fig. 1. Simulation of the controlled system without observer ($k_v = 12$).

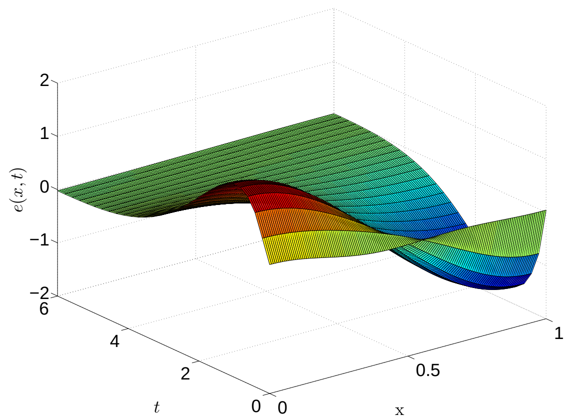

Fig. 2. Simulation of the observer equation ($a = 12, b = 0.8$).

where the coefficients $d_i, (i = 0, 1, 2, 3)$ are given by

$$d_0 = 0.25, \quad (65)$$

$$d_1 = 0.25a - 0.50b + 4.84, \quad (66)$$

$$d_2 = 7.34a - 9.74b - 74.33, \quad (67)$$

$$d_3 = -.96a + 147.39b + 9.61. \quad (68)$$

By using the Hurwitz stability criterion, we can obtain the stability condition with respect to (a, b):

$$d_i > 0, \ (i = 1, 2, 3), \ \text{ and } d_1 d_2 > d_0 d_3. \quad (69)$$

For the numerical simulation, we choose $I_N = 4$ and compute the matrices defined in (48)–(52). We choose $k_v = 12, a = 12, b = 0.8$ to satisfy the stability conditions (63)–(69). When $f_h = 0$, i.e., there is no truncation error for the input function $f(x)$ involved in the control design, the closed–loop dynamics is shown in Fig. 1, and the observer dynamics is shown in Fig. 2. The observer–based feedback system dynamics is shown in Fig. 3.

V. Conclusions

Sturm–Liouville theory and numerical spectral analysis of differential operators are used in this work to approach the stabilization problem of an unstable parabolic PDE with constant diffusion coefficient. This method reduces the control synthesis for linear PDE systems to a parametric stabilization problem for a Sturm-Liouville system, which is

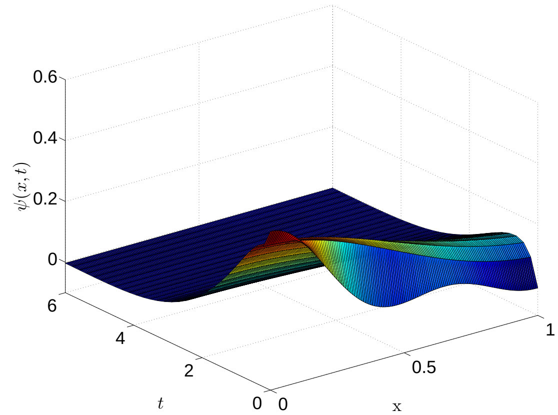

Fig. 3. Simulation of the controlled system with observer.

solved using the finite dimensional truncation approach based on the pseudo-spectral method. The design of a state observer based on a boundary measurement is also approached in this work. Analytical and numerical work is carried out for the solution of the Sturm-Liouville system arising during the observer design in terms of three different scenarios for the observer gain: constant, harmonic and general gains. The analysis concludes that it is required to have harmonic components in the observer gains instead of pure constants. A numerical algorithm using the pseudo-spectral method is proposed for the observer design with general gain.

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