Backstepping Control of Density and Energy Profiles in a Burning Tokamak Plasma

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Abstract— **The control of plasma density profiles is one of the most fundamental problems in fusion reactors. During reactor operation, the spatial profiles of deuterium-tritium fuel, alpha-particles generated by fusion reactions, and energy must be precisely regulated. Here we apply a backstepping boundary control technique to stabilize an unstable equilibrium in a burning plasma. A one-dimensional approximation of the transport equation for energy as well as the density of deuterium-tritium fuel ions and alpha-particles is represented in cylindrical coordinates by a system of partial differential equations (PDEs). To control the ion and energy density profiles, the PDE system is discretized in space using a finite difference method and a backstepping design is applied to obtain a discrete transformation from the original system into an asymptotically stable target system. Numerical simulations of the resulting boundary control law show that the profiles can be successfully controlled with just one step of backstepping.**

I. Introduction

To realize the promise of nuclear fusion and make it an economical energy source, tokamak reactors may need to be operated at inherently unstable operating points and steady state operation will require precise control of kinetic and magnetic variables within the fusion plasma. Most approaches to the control of kinetic variables in tokamaks begin by considering 0-D (zero-dimensional) models of transport within the fusion plasma. In these models, quantities are spatially averaged over the volume of the plasma, simplifying the governing equations to ordinary differential equations (ODEs) and allowing the problem to be approached with lumped-parameter control design techniques. The problem is often simplified further by linearizing the nonlinear 0-D model and putting the model in a standard control form for which linear control techniques can be used. In [1], [2], the linearization of the model is avoided and much higher levels of performance and robustness are achieved. However, the 0-D control of the system using modulation of bulk heating, fueling and impurity density does not take into account the 1-D (one-dimensional) effect of this modulation on the spatial profiles. In a reactor, the heating, fueling, and impurity injection rates are indeed distributed throughout the plasma and affect the shape of the kinetic variable profiles.

The importance of these profiles stems from their effect on transport, confinement times, and magnetohydrodynamic stability within the fusion plasma. A reliable profile control system will be essential for creating and maintaining kinetic profiles that minimize transport and maximize reactor performance within stability limits. Previous work in the field, [3], [4], [5], [6], [7] has recognized the importance of kinetic profile control in tokamak reactors. In these pieces of work, a 1-D plasma model is represented by a set of partial differential equations (PDEs) and various methods are utilized to reduce the distributed parameter model to a lumped-parameter one. The resulting set of ODEs are then linearized and conventional linear control techniques are used for controller design.

In contrast to this previous body of work, the control method presented in [8] was based on the full non-linear model of density and temperature in a non-burning plasma. By accounting for the non-linear terms, this approach avoids the operability limits created by linearization. The backstepping control technique used has been successfully demonstrated for other physical applications in work such as [9] and [10]. In [11], we applied the backstepping technique to controlling and tracking electron density and effective atomic number in a non-burning plasma. In this work, we apply this technique to the control of density and energy profiles in a burning plasma.

The paper is organized as follows. In Section II the one-dimensional burning plasma model is introduced. In Section III the control objective and actuation methods are outlined. Section IV shows the backstepping technique used and contains an analysis of the stability of the target system. Simulation results showing successful stabilization of an unstable set of equilibrium profiles are contained in Section V. Concluding remarks and a discussion of future work are given in Section VI.

II. One-Dimensional Burning Plasma Model

The one-dimensional burning plasma model must include the dynamics of the spatial profiles of the density of α-particles, the deuterium-tritium fuel, as well as the spatial profile of the energy in the system. The model used in this work is based on standard 1-D transport equations. The model is simplified by assuming a constant diffusivity and a negligible pinch velocity. The equations for particle densities and plasma energy are:

$$\frac{\partial n_\alpha}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial n_\alpha}{\partial r}\right) + \left(\frac{n_{DT}}{2}\right)^2 \langle\sigma v\rangle \quad (1)$$

$$\frac{\partial n_{DT}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial n_{DT}}{\partial r}\right) - 2\left(\frac{n_{DT}}{2}\right)^2 \langle\sigma v\rangle + S_{DT} \quad (2)$$

$$\frac{\partial E}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial E}{\partial r}\right) + Q_\alpha \left(\frac{n_{DT}}{2}\right)^2 \langle\sigma v\rangle - P_{rad} + P_{aux} \quad (3)$$

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where $\langle \sigma v \rangle$ is the DT reactivity, S_{DT} is the DT fuel injection, and $Q_\alpha = 3.52$ MeV is the alpha particle energy. P_{aux} and P_{rad} represent the auxiliary power and radiation losses, respectively.

The DT reactivity $\langle \sigma v \rangle$ is a highly nonlinear, positive, and bounded function of the plasma temperature T and is calculated by

$$\langle \sigma v \rangle = exp\left(\frac{a_1}{T^r} + a_2 + a_3 T + a_4 T^2 + a_5 T^3 + a_6 T^4\right) \quad (4)$$

where the parameters a_i and r are taken from [12]. The plasma temperature is a function of the energy and the total plasma density, i.e., $T = \frac{2}{3}\frac{E}{n}$, and the total plasma density is given by the sum of the ion and electron densities, n_i and n_e:

$$n_i = n_{DT} + n_\alpha \quad (5)$$

$$n_e = n_{DT} + 2n_\alpha \quad (6)$$

$$n = n_i + n_e = 2n_{DT} + 3n_\alpha \quad (7)$$

The radiation loss P_{rad} considered in this work is given by

$$P_{rad} = P_{brem} = A_b Z_{eff} n_e^2 \sqrt{T} \quad (8)$$

where $A_b = 5.5 \times 10^{-37}$ $\mathrm{Wm^3}/\sqrt{\mathrm{keV}}$ is the bremsstrahlung radiation coefficient, Z_{eff} is the effective atomic number, and n_e is the electron density. Note that the control design presented in this work could easily be extended to include other forms of radiation losses and this choice of model is only used for simplification of presentation. The effective atomic number is given by

$$Z_{eff} = \sum_i \frac{n_i Z_i^2}{n_e} = \frac{n_{DT} + 4n_\alpha}{n_e} \quad (9)$$

In this work, the following boundary conditions are used:

$$\left.\frac{\partial n_\alpha}{\partial r}\right|_{r=0} = \left.\frac{\partial n_{DT}}{\partial r}\right|_{r=0} = \left.\frac{\partial E}{\partial r}\right|_{r=0} = 0 \quad (10)$$

$$\left.\frac{\partial n_\alpha}{\partial r}\right|_{r=a} = -k_\alpha n_\alpha(a) \quad (11)$$

$$\left.\frac{\partial n_{DT}}{\partial r}\right|_{r=a} = -k_{DT} n_{DT}(a) \quad (12)$$

$$\left.\frac{\partial E}{\partial r}\right|_{r=a} = -k_E E(a) \quad (13)$$

where k_α, k_{DT}, and k_E are positive and a is the radius of the plasma.

III. Control Objective

At equilibrium, the DT fuel, alpha particle, and energy densities are no longer changing with respect to time and the model simplifies to a set of ODEs with respect to the space coordinate

$$0 = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial \bar{n}_\alpha}{\partial r}\right) + \bar{S}_\alpha \quad (14)$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial \bar{n}_{DT}}{\partial r}\right) - 2\bar{S}_\alpha + \bar{S}_{DT} \quad (15)$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial \bar{E}}{\partial r}\right) + Q_\alpha \bar{S}_\alpha - \bar{P}_{rad} + \bar{P}_{aux} \quad (16)$$

where we have written the alpha particle generation as $S_\alpha = \left(\frac{n_{DT}}{2}\right)^2 \langle \sigma v \rangle$ and we use upper bar notation to represent the equilibrium value of a variable. The equilibrium profiles are determined by the equilibrium fueling and auxiliary heating profiles.

In this work, actuation is only considered at the plasma's edge and the fueling and heating rates are used only to define the equilibrium profiles. Writing $n_\alpha(r,t) = \bar{n}_\alpha(r) + \tilde{n}_\alpha(r,t)$, $n_{DT}(r,t) = \bar{n}_{DT}(r) + \tilde{n}_{DT}(r,t)$, $E(r,t) = \bar{E}(r) + \tilde{E}(r,t)$, $S_\alpha(r,t) = \bar{S}_\alpha(r) + \tilde{S}_\alpha(r,t)$, $S_{DT}(r,t) = \bar{S}_{DT}(r)$, $P_{rad}(r,t) = \bar{P}_{rad}(r) + \tilde{P}_{rad}(r,t)$, and $P_{aux}(r,t) = \bar{P}_{aux}(r)$, the dynamics of the deviation variables $\tilde{n}_\alpha(r,t)$, $\tilde{n}_{DT}(r,t)$, and $\tilde{E}(r,t)$ are given by

$$\frac{\partial \tilde{n}_\alpha}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial(\tilde{n}_\alpha + \bar{n}_\alpha)}{\partial r}\right) + \tilde{S}_\alpha + \bar{S}_\alpha \quad (17)$$

$$\frac{\partial \tilde{n}_{DT}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial(\tilde{n}_{DT} + \bar{n}_{DT})}{\partial r}\right) - 2\tilde{S}_\alpha - 2\bar{S}_\alpha + \bar{S}_{DT} \quad (18)$$

$$\frac{\partial \tilde{E}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial(\tilde{E} + \bar{E})}{\partial r}\right) + Q_\alpha \tilde{S}_\alpha + Q_\alpha \bar{S}_\alpha - \tilde{P}_{rad} - \bar{P}_{rad} + \bar{P}_{aux} \quad (19)$$

Noting (14), (15), and (16) and taking into account that

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial(\cdot)}{\partial r}\right] = \frac{\partial}{\partial r}\left[D\frac{\partial(\cdot)}{\partial r}\right] + \frac{1}{r}D\frac{\partial(\cdot)}{\partial r}$$

the equations can be rewritten as

$$\frac{\partial \tilde{n}_\alpha}{\partial t} = D\frac{\partial^2 \tilde{n}_\alpha}{\partial r^2} + \frac{1}{r}D\frac{\partial \tilde{n}_\alpha}{\partial r} + \tilde{S}_\alpha \quad (20)$$

$$\frac{\partial \tilde{n}_{DT}}{\partial t} = D\frac{\partial^2 \tilde{n}_{DT}}{\partial r^2} + \frac{1}{r}D\frac{\partial \tilde{n}_{DT}}{\partial r} - 2\tilde{S}_\alpha \quad (21)$$

$$\frac{\partial \tilde{E}}{\partial t} = D\frac{\partial^2 \tilde{E}}{\partial r^2} + \frac{1}{r}D\frac{\partial \tilde{E}}{\partial r} + Q_\alpha \tilde{S}_\alpha - \tilde{P}_{rad} \quad (22)$$

Recall that $\tilde{S}_\alpha$ and $\tilde{P}_{rad}$ are nonlinear functions of all of the states. The boundary conditions are written as

$$\left.\frac{\partial \tilde{n}_\alpha}{\partial r}\right|_{r=0} = \left.\frac{\partial \tilde{n}_{DT}}{\partial r}\right|_{r=0} = \left.\frac{\partial \tilde{E}}{\partial r}\right|_{r=0} = 0 \quad (23)$$

$$\left.\frac{\partial \tilde{n}_\alpha}{\partial r}\right|_{r=a} = -k_\alpha \tilde{n}_\alpha(a) + \Delta(\tilde{n}_\alpha)_r \quad (24)$$

$$\left.\frac{\partial \tilde{n}_{DT}}{\partial r}\right|_{r=a} = -k_{DT} \tilde{n}_{DT}(a) + \Delta(\tilde{n}_{DT})_r \quad (25)$$

$$\left.\frac{\partial \tilde{E}}{\partial r}\right|_{r=a} = -k_E \tilde{E}(a) + \Delta\tilde{E}_r \quad (26)$$

The objective of the controller is to force $\tilde{n}_\alpha(r,t)$, $\tilde{n}_{DT}(r,t)$ and $\tilde{E}(r,t)$ to zero using $\Delta(\tilde{n}_\alpha)_r$, $\Delta(\tilde{n}_{DT})_r$, and $\Delta\tilde{E}_r$ as actuation at the plasma's edge.

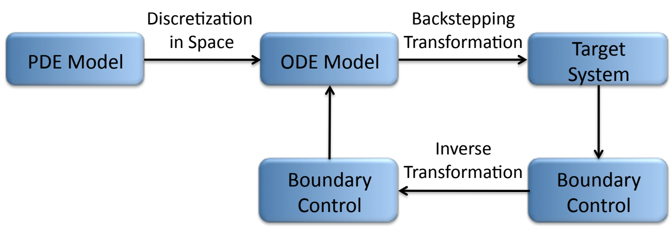

Fig. 1. Controller scheme.

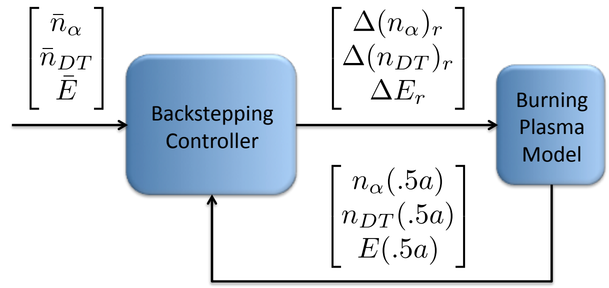

Fig. 2. Block diagram of simulation process.

IV. Backstepping Technique

A backstepping technique is used to transform the original system of equations into an asymptotically stable target system. Figure 1 illustrates the technique.

By defining $h = \frac{1}{N}$, where N is an integer, and using the notation $x^i(t) = x(ih,t)$, $i = 0,1,...,N$, the discretized version of (20) - (22) can be written as

$$\dot{\tilde{n}}_\alpha^i = D\frac{\tilde{n}_\alpha^{i+1} - 2\tilde{n}_\alpha^i + \tilde{n}_\alpha^{i-1}}{h^2} + \frac{D}{ih}\frac{\tilde{n}_\alpha^{i+1} - \tilde{n}_\alpha^i}{h} + \tilde{S}_\alpha^i \tag{27}$$

$$\dot{\tilde{n}}_{DT}^i = D\frac{\tilde{n}_{DT}^{i+1} - 2\tilde{n}_{DT}^i + \tilde{n}_{DT}^{i-1}}{h^2} + \frac{D}{ih}\frac{\tilde{n}_{DT}^{i+1} - \tilde{n}_{DT}^i}{h} - 2\tilde{S}_\alpha^i \tag{28}$$

$$\dot{\tilde{E}}^i = D\frac{\tilde{E}^{i+1} - 2\tilde{E}^i + \tilde{E}^{i-1}}{h^2} + \frac{D}{ih}\frac{\tilde{E}^{i+1} - \tilde{E}^i}{h} + Q_\alpha\tilde{S}_\alpha^i - \tilde{P}_{rad}^i \tag{29}$$

with the boundary conditions written as

$$\frac{\tilde{n}_\alpha^1 - \tilde{n}_\alpha^0}{h} = \frac{\tilde{n}_{DT}^1 - \tilde{n}_{DT}^0}{h} = \frac{\tilde{E}^1 - \tilde{E}^0}{h} = 0 \tag{30}$$

$$\frac{\tilde{n}_\alpha^N - \tilde{n}_\alpha^{N-1}}{h} = -k_\alpha\tilde{n}_\alpha^N + \Delta(\tilde{n}_\alpha)_r \tag{31}$$

$$\frac{\tilde{n}_{DT}^N - \tilde{n}_{DT}^{N-1}}{h} = -k_{DT}\tilde{n}_{DT}^N + \Delta(\tilde{n}_{DT})_r \tag{32}$$

$$\frac{\tilde{E}^N - \tilde{E}^{N-1}}{h} = -k_E\tilde{E}^N + \Delta\tilde{E}_r \tag{33}$$

Next, an asymptotically stable target system is considered

$$\frac{\partial\tilde{w}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial\tilde{w}}{\partial r}\right] - C_w\tilde{w} \tag{34}$$

$$= D\frac{\partial^2\tilde{w}}{\partial r^2} + \frac{1}{r}D\frac{\partial\tilde{w}}{\partial r} - C_w\tilde{w} \tag{35}$$

$$\frac{\partial\tilde{m}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial\tilde{m}}{\partial r}\right] - C_m\tilde{m} \tag{36}$$

$$= D\frac{\partial^2\tilde{m}}{\partial r^2} + \frac{1}{r}D\frac{\partial\tilde{m}}{\partial r} - C_m\tilde{m} \tag{37}$$

$$\frac{\partial\tilde{F}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial\tilde{F}}{\partial r}\right] - C_F\tilde{F} \tag{38}$$

$$= D\frac{\partial^2\tilde{F}}{\partial r^2} + \frac{1}{r}D\frac{\partial\tilde{F}}{\partial r} - C_F\tilde{F} \tag{39}$$

with $C_w, C_m, C_F > 0$ and the following boundary conditions

$$\left.\frac{\partial\tilde{w}}{\partial r}\right|_{r=0} = \left.\frac{\partial\tilde{m}}{\partial r}\right|_{r=0} = \left.\frac{\partial\tilde{F}}{\partial r}\right|_{r=0} = 0 \tag{40}$$

$$\left.\frac{\partial\tilde{w}}{\partial r}\right|_{r=a} = -G\tilde{w}(a) \tag{41}$$

$$\left.\frac{\partial\tilde{m}}{\partial r}\right|_{r=a} = -G\tilde{m}(a) \tag{42}$$

$$\left.\frac{\partial\tilde{F}}{\partial r}\right|_{r=a} = -G\tilde{F}(a) \tag{43}$$

with $G > 0$. The choice of the target system is motivated by the need to maintain the parabolic character of the partial differential equation (to keep the highest order derivatives) while decoupling and stabilizing the system by removing the problematic terms. The target system can be discretized as

$$\dot{\tilde{w}}^i = D\frac{\tilde{w}^{i+1} - 2\tilde{w}^i + \tilde{w}^{i-1}}{h^2} + \frac{1}{ih}D\frac{\tilde{w}^{i+1} - \tilde{w}^i}{h} - C_w\tilde{w} \tag{44}$$

$$\dot{\tilde{m}}^i = D\frac{\tilde{m}^{i+1} - 2\tilde{m}^i + \tilde{m}^{i-1}}{h^2} + \frac{1}{ih}D\frac{\tilde{m}^{i+1} - \tilde{m}^i}{h} - C_m\tilde{m} \tag{45}$$

$$\dot{\tilde{F}}^i = D\frac{\tilde{F}^{i+1} - 2\tilde{F}^i + \tilde{F}^{i-1}}{h^2} + \frac{1}{ih}D\frac{\tilde{F}^{i+1} - \tilde{F}^i}{h} - C_F\tilde{F} \tag{46}$$

with the boundary conditions written as

$$\frac{\tilde{w}_1 - \tilde{w}_0}{h} = \frac{\tilde{m}^1 - \tilde{m}^0}{h} = \frac{\tilde{F}^1 - \tilde{F}^0}{h} = 0 \tag{47}$$

$$\frac{\tilde{w}^N - \tilde{w}^{N-1}}{h} = -G\tilde{w}^N \tag{48}$$

$$\frac{\tilde{m}^N - \tilde{m}^{N-1}}{h} = -G\tilde{m}^N \tag{49}$$

$$\frac{\tilde{F}^N - \tilde{F}^{N-1}}{h} = -G\tilde{F}^N \tag{50}$$

Next, a backstepping transformation is found in the form

$$\tilde{w}^i = \tilde{n}_\alpha^i - \omega^{i-1}(\tilde{E}^0,\dots,\tilde{E}^{i-1},\tilde{n}_{DT}^0,\dots,\tilde{n}_{DT}^{i-1},\tilde{n}_\alpha^0,\dots,\tilde{n}_\alpha^{i-1}) \tag{51}$$

$$\tilde{m}^i = \tilde{n}_{DT}^i - \beta^{i-1}(\tilde{E}^0,\dots,\tilde{E}^{i-1},\tilde{n}_{DT}^0,\dots,\tilde{n}_{DT}^{i-1},\tilde{n}_\alpha^0,\dots,\tilde{n}_\alpha^{i-1}) \tag{52}$$

$$\tilde{F}^i = \tilde{E}^i - \alpha^{i-1}(\tilde{E}^0,\dots,\tilde{E}^{i-1},\tilde{n}_{DT}^0,\dots,\tilde{n}_{DT}^{i-1},\tilde{n}_\alpha^0,\dots,\tilde{n}_\alpha^{i-1}) \tag{53}$$

By subtracting (44) from (27) ((45) from (28), (46) from (29)), the expression $\dot{\omega}^{i-1} = \dot{\tilde{n}}_\alpha^i - \dot{\tilde{w}}^i$ ($\dot{\beta}^{i-1} = \dot{\tilde{n}}_{DT}^i - \dot{\tilde{m}}^i$, $\dot{\alpha}^{i-1} = \dot{\tilde{E}}^i - \dot{\tilde{F}}^i$) is obtained, which can be put in terms of $\omega^{k-1} = \tilde{n}_\alpha^k - \tilde{w}^k$, $k = i-1, i, i+1$ ($\beta^{k-1} = \tilde{n}_{DT}^k - \tilde{m}^k$, $k = i-1, i, i+1$, $\alpha^{k-1} = \tilde{E}^k - \tilde{F}^k$, $k = i-1, i, i+1$) and rearranged to obtain

$$\omega^i = \frac{1}{D+D/i}\left[\left(2D+\frac{D}{i}+C_w h^2\right)\omega^{i-1} - D\omega^{i-2} - h^2 C_w \tilde{n}_\alpha^i + h^2\dot{\omega}^{i-1} - h^2\tilde{S}_\alpha^i\right] \quad (54)$$

$$\beta^i = \frac{1}{D+D/i}\left[\left(2D+\frac{D}{i}+C_m h^2\right)\beta^{i-1} - D\beta^{i-2} - h^2 C_m \tilde{n}_{DT}^i + h^2\dot{\beta}^{i-1} + 2h^2\tilde{S}_\alpha^i\right] \quad (55)$$

$$\alpha^i = \frac{1}{D+D/i}\left[\left(2D+\frac{D}{i}+C_F h^2\right)\alpha^{i-1} - D\alpha^{i-2} - h^2 C_F \tilde{E}^i + h^2\dot{\alpha}^{i-1} - h^2 Q_\alpha\tilde{S}_\alpha^i + h^2\tilde{P}_{rad}^i\right] \quad (56)$$

where $\omega^0 = \beta^0 = \alpha^0 = 0$ and $\dot{\omega}^{i-1}$, $\dot{\beta}^{i-1}$, and $\dot{\alpha}^{i-1}$ are calculated as

$$\dot{\omega}^{i-1} = \sum_{k=1}^{i-1}\frac{\partial\omega^{i-1}}{\partial\tilde{n}_{DT}^k}\dot{\tilde{n}}_{DT}^k + \sum_{k=1}^{i-1}\frac{\partial\omega^{i-1}}{\partial\tilde{E}^k}\dot{\tilde{E}}^k + \sum_{k=1}^{i-1}\frac{\partial\omega^{i-1}}{\partial\tilde{n}_\alpha^k}\dot{\tilde{n}}_\alpha^k \quad (57)$$

$$\dot{\beta}^{i-1} = \sum_{k=1}^{i-1}\frac{\partial\beta^{i-1}}{\partial\tilde{n}_{DT}^k}\dot{\tilde{n}}_{DT}^k + \sum_{k=1}^{i-1}\frac{\partial\beta^{i-1}}{\partial\tilde{E}^k}\dot{\tilde{E}}^k + \sum_{k=1}^{i-1}\frac{\partial\beta^{i-1}}{\partial\tilde{n}_\alpha^k}\dot{\tilde{n}}_\alpha^k \quad (58)$$

$$\dot{\alpha}^{i-1} = \sum_{k=1}^{i-1}\frac{\partial\alpha^{i-1}}{\partial\tilde{n}_{DT}^k}\dot{\tilde{n}}_{DT}^k + \sum_{k=1}^{i-1}\frac{\partial\alpha^{i-1}}{\partial\tilde{E}^k}\dot{\tilde{E}}^k + \sum_{k=1}^{i-1}\frac{\partial\alpha^{i-1}}{\partial\tilde{n}_\alpha^k}\dot{\tilde{n}}_\alpha^k \quad (59)$$

Next, subtracting (48) from (31) ((49) from (32), (50) from (33)) and putting the resulting equation in terms of $\omega^{k-1} = \tilde{n}_\alpha^k - \tilde{w}^k$, $k = i-1, i$ ($\beta^{k-1} = \tilde{n}_{DT}^k - \tilde{m}^k$, $k = i-1, i$, $\alpha^{k-1} = \tilde{E}^k - \tilde{F}^k$, $k = i-1, i$) the control $\Delta(\tilde{n}_\alpha)_r$ ($\Delta(\tilde{n}_{DT})_r$, $\Delta\tilde{E}_r$) can be defined as

$$\Delta(\tilde{n}_\alpha)_r = \frac{\omega^{N-1}-\omega^{N-2}}{h} - k_\alpha\tilde{n}_\alpha^N - G(\tilde{n}_\alpha^N - \omega^{N-1}) \quad (60)$$

$$\Delta(\tilde{n}_{DT})_r = \frac{\beta^{N-1}-\beta^{N-2}}{h} - k_{DT}\tilde{n}_{DT}^N - G(\tilde{n}_{DT}^N - \beta^{N-1}) \quad (61)$$

$$\Delta\tilde{E}_r = \frac{\alpha^{N-1}-\alpha^{N-2}}{h} - k_E\tilde{E}^N - G(\tilde{E}^N - \alpha^{N-1}) \quad (62)$$

These equations can then be rewritten as the stabilizing laws for the α-particle density, DT density, and energy at the plasma's edge:

$$\tilde{n}_\alpha^N = \omega^{N-1} + \frac{1}{(1+Gh)}\left[\tilde{n}_\alpha^{N-1} - \omega^{N-2}\right] \quad (63)$$

$$\tilde{n}_{DT}^N = \beta^{N-1} + \frac{1}{(1+Gh)}\left[\tilde{n}_{DT}^{N-1} - \beta^{N-2}\right] \quad (64)$$

$$\tilde{E}^N = \alpha^{N-1} + \frac{1}{(1+Gh)}\left[\tilde{E}^{N-1} - \alpha^{N-2}\right] \quad (65)$$

To show the asymptotic stability of the target system, we consider the Lyapunov function candidate

$$V = \frac{1}{2}\int_0^a r\left(\tilde{w}^2 + \tilde{m}^2 + \tilde{F}^2\right)dr$$

Taking the derivative of this function with respect to time gives

$$\begin{aligned}
\dot{V} &= \int_0^a r\left(\tilde{w}\dot{\tilde{w}} + \tilde{m}\dot{\tilde{m}} + \tilde{F}\dot{\tilde{F}}\right)dr \\
&= \int_0^a r\left(\tilde{w}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial\tilde{w}}{\partial r}\right] - C_w\tilde{w}\right\}\right. \\
&\quad + \tilde{m}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial\tilde{m}}{\partial r}\right] - C_m\tilde{m}\right\} \\
&\quad \left. + \tilde{F}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial\tilde{F}}{\partial r}\right] - C_F\tilde{F}\right\}\right)dr \\
&= \tilde{w}rD\frac{\partial\tilde{w}}{\partial r}\bigg|_0^a - \int_0^a rD\left(\frac{\partial\tilde{w}}{\partial r}\right)^2 dr - C_w\int_0^a r\tilde{w}^2 dr \\
&\quad + \tilde{m}rD\frac{\partial\tilde{m}}{\partial r}\bigg|_0^a - \int_0^a rD\left(\frac{\partial\tilde{m}}{\partial r}\right)^2 dr - C_m\int_0^a r\tilde{m}^2 dr \\
&\quad + \tilde{F}rD\frac{\partial\tilde{F}}{\partial r}\bigg|_0^a - \int_0^a rD\left(\frac{\partial\tilde{F}}{\partial r}\right)^2 dr - C_F\int_0^a r\tilde{F}^2 dr \\
&= aD\tilde{w}(a)\tilde{w}_r(a) - \int_0^a rD\tilde{w}_r^2 dr - C_w\int_0^a r\tilde{w}^2 dr \\
&\quad + aD\tilde{m}(a)\tilde{m}_r(a) - \int_0^a rD\tilde{m}_r^2 dr - C_m\int_0^a r\tilde{m}^2 dr \\
&\quad + aD\tilde{F}(a)\tilde{F}_r(a) - \int_0^a rD\tilde{F}_r^2 dr - C_F\int_0^a r\tilde{F}^2 dr
\end{aligned}$$

where the operation $\frac{\partial(\cdot)}{\partial r}$ is denoted by $(\cdot)_r$. Using the boundary conditions (41) through (43), this can be written as

$$\begin{aligned}
\dot{V} = &-GaD\tilde{w}^2(a) - C_w\int_0^a r\tilde{w}^2 dr - \int_0^a rD\tilde{w}_r^2 dr \\
&- GaD\tilde{m}^2(a) - C_m\int_0^a r\tilde{m}^2 dr - \int_0^a rD\tilde{m}_r^2 dr \\
&- GaD\tilde{F}^2(a) - C_F\int_0^a r\tilde{F}^2 dr - \int_0^a rD\tilde{F}_r^2 dr
\end{aligned}$$

It can be concluded that

$$\begin{aligned}
\dot{V} \le &-CV - GaD\tilde{w}^2(a) - \int_0^a rD\tilde{w}_r^2 dr - GaD\tilde{m}^2(a) \\
&- \int_0^a rD\tilde{m}_r^2 dr - GaD\tilde{F}^2(a) - \int_0^a rD\tilde{F}_r^2 dr
\end{aligned}$$

and, because the last six terms in the expression are all negative semi-definite we can conclude that $\dot{V} \le -CV$, proving the asymptotic stability of the target system.

The control strategy is summarized in Figure 2. First, a desired set of equilibrium profiles is determined. These profiles are then used as references by the backstepping controller, which actuate the ion densities and energy at the plasma edge in order to achieve the desired profile shapes and spatial averages for the alpha particle, DT fuel, and energy density. We will show through simulations in the next section that measures of the profiles at $r = 0.5a$ are enough to regulate the plasma profiles.

V. Simulations

The discretized burning plasma was simulated using an implicit finite difference scheme. The results shown are for

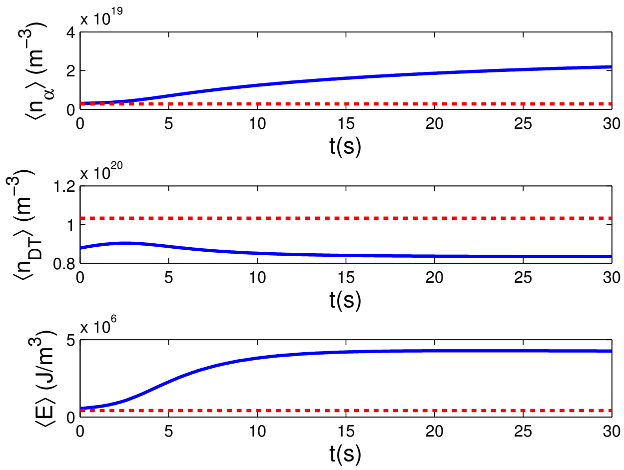

Fig. 3. Spatial averages of the open-loop (uncontrolled) profiles. The desired equilibrium values are shown as red dotted lines. The solid blue lines show the simulated evolution of the system.

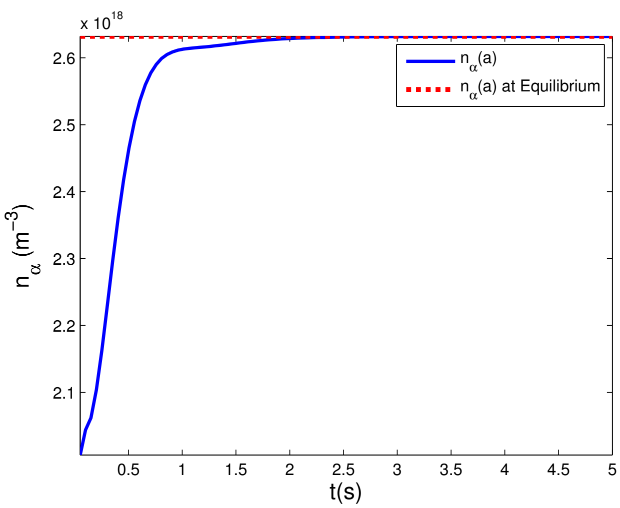

Fig. 4. Edge modulation for α-particle density.

an equilibrium described by the spatial averages, diffusivity, and boundary conditions in Table I and using a particular set of fueling and heating profiles. Figure 3 shows that without control, the chosen equilibrium is unstable. In this case, the initial perturbations in the shape and magnitude cause a thermal excursion and the system comes to rest at a higher temperature equilibrium. In other cases, initial perturbations may cause quenching, in which the temperature drops. Closed-loop simulation of the system with the same initial perturbations shows that the proposed controller is able to reject the initial perturbations and return the system to the desired equilibrium. Figures 4 through 6 show the controller's modulation of the α-particle density, DT density, and energy at the edge of the plasma. Figures 7 through 9 show the effect of this modulation on the time evolution of the error in the α-particle, DT, and energy density profiles, respectively. Figure 10 shows how the spatial averages of the α-particles, DT ions, and energy return to the desired equilibrium values.

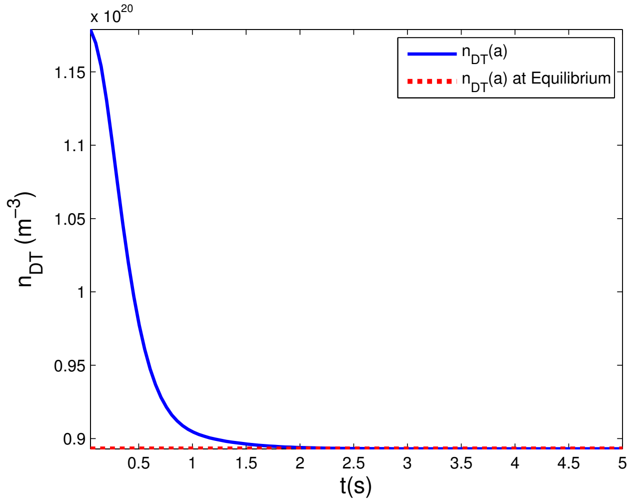

Fig. 5. Edge modulation for DT fuel density.

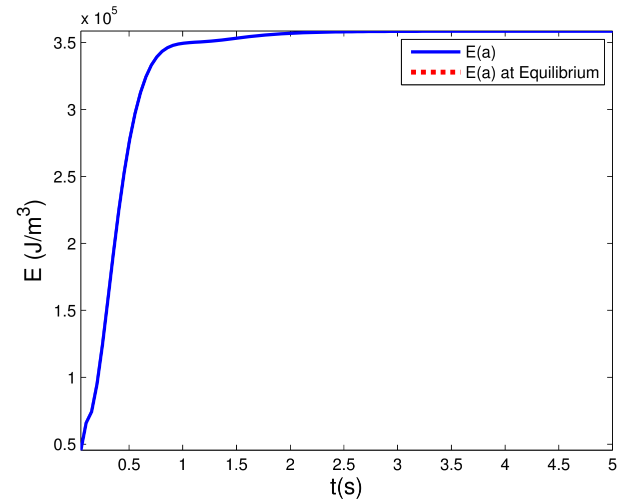

Fig. 6. Edge modulation for energy.

TABLE I

Simulation Parameters

Parameter	Value	Parameter	Value
$\langle\bar{n}_\alpha\rangle$	2.77×10^{18}	$\langle\bar{n}_{DT}\rangle$	1.03×10^{20}
$\langle\bar{E}\rangle$	4.20×10^{05}	D	0.8
k_α	0.085	k_{DT}	0.210
k_E	0.287		

VI. Conclusions and Future Work

A non-linear feedback controller based on Lyapunov backstepping that achieves asymptotic stabilization of the equilibrium ion and energy density profiles in a cylindrical burning plasma has been designed. The controller uses actuation of the α-particle, energy, and DT ion fluxes at the plasma's edge to stabilize the respective profiles. The resulting controller holds for any finite discretization in space of the original PDE model and the simulation in this work shows that a controller using just one step of backstepping, or just one measurement from inside the plasma, successfully

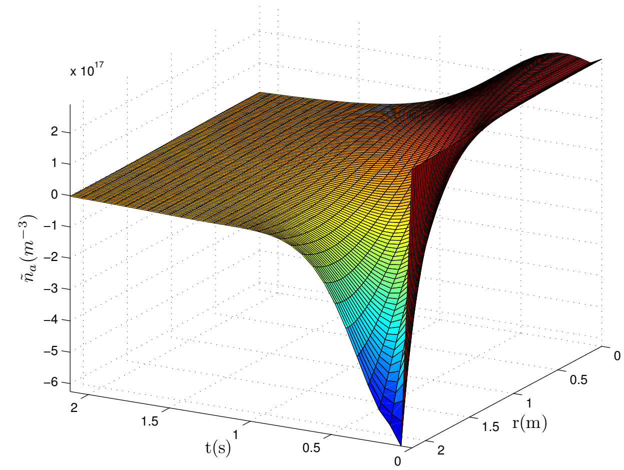

Fig. 7. Time evolution of the α-particle density profile error.

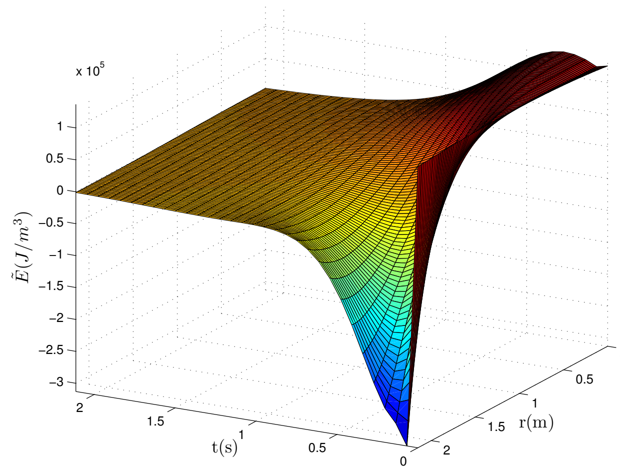

Fig. 9. Time evolution of the energy profile error.

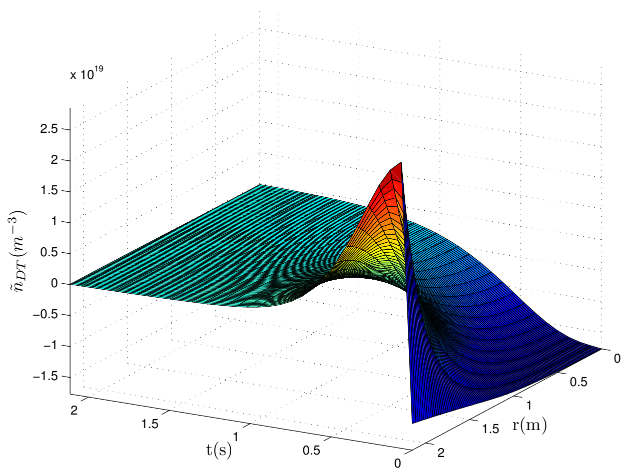

Fig. 8. Time evolution of the DT density profile error.

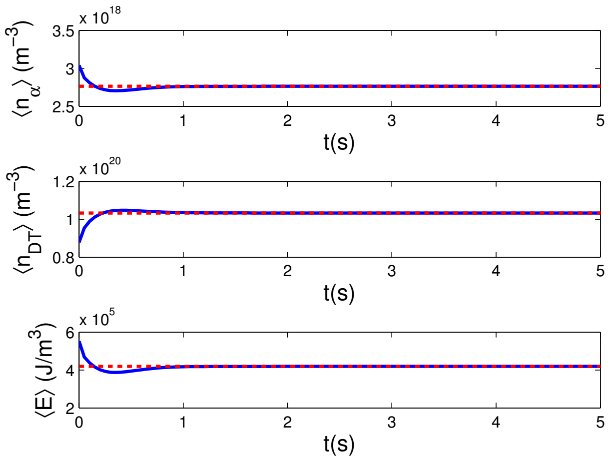

Fig. 10. Closed loop spatial averages. Desired values in red dotted lines.

controls the density profiles. The feasibility of controlling kinetic profiles in a burning plasma using boundary control techniques has been shown. However, more study will be necessary to find physical methods for the modulation of the kinetic variables at the edge of the plasma, i.e. achieving the desired values of $\Delta(\tilde{n}_\alpha)_r$, $\Delta(\tilde{n}_{DT})_r$, and $\Delta\tilde{E}_r$. This will have to be done through modulation of the physical properties of the scrape-off layer (SOL) such as gas puffing, gas pumping, or impurity injection. Since the major difficulty is in modulation of $\Delta\tilde{E}_r$, one approach will be the development of a similar control scheme that avoids boundary actuation of the energy and instead considers modulation of the heating in the core of the plasma. Moving forward, model improvements will be made by including models for the diffusivity and pinch velocity, as well as models of the SOL in order to apply more realistic boundary conditions to the system.

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