On the Stability of Receding Horizon Control of Bilinear Parabolic PDE Systems

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Abstract— **We propose a framework to solve an optimal control problem for a bilinear parabolic partial differential equation (PDE). We formulate the problem as an abstract bilinear-quadratic regulator (BQR) problem. A receding horizon control (RHC) algorithm to solve the problem based on the infinite-dimensional system is proposed and stability of the algorithm for the solution of the BQR problem is studied. A successive approximation approach is used to numerically solve the quadratic optimal control problem subject to the bilinear PDE model associated with the RHC scheme. Finally, the proposed approach is applied to the current profile control problem in tokamak plasmas and its effectiveness is demonstrated in simulations.**

I. INTRODUCTION

The control of linear or quasi-linear parabolic diffusion-reaction partial differential equations (PDE) has been extensively studied using *interior control* (see [1] and references therein) or *boundary control* (see [2] and references therein). Recently, the control of bilinear parabolic partial differential equations via actuation of the diffusive coefficient term, named *diffusivity control* here, has caught increasing interest. The diffusive coefficient term in a parabolic PDE is not necessary fixed or uncontrollable. For example, diffusivity regulation arises in the control of the current density profile in magnetically confined fusion plasmas [3], where physical actuators such as plasma total current, line-averaged density and non-inductive total power are used to steer the plasma current density to a desired profile in a designated time period. By modulating these physical actuators it is possible not only to vary the amount of non-inductive current driven into the system (interior control) and the total plasma current (boundary control) but also to modify the resistivity of the plasma (diffusivity control).

Receding horizon control (RHC), also named as model predictive control (MPC), has become an attractive feedback strategy. In the past two decades, numerous results have been published on receding horizon control for finite-dimensional systems. In recent years, some methods have been proposed to deal with infinite-dimensional systems. Ito and Kunisch in [4] provided a general framework to control infinite-dimensional systems using a RHC scheme with guaranteed stability. Model reduction by inertial manifold theory and partition of the eigenspectrum of the PDE operator has been proposed by Christofides and coworkers [5], and a RHC scheme aiming at dealing with quasi-linear parabolic PDE systems with control and state constrains is presented in [6]. In [7], Benner and coworkers have obtained some new results in using RHC/LQG-based optimal control of an infinite-dimensional reaction-diffusion system.

Motivated by the increasing interest in bilinear parabolic PDE control, we focus in this work on the development of a framework for the design of a receding horizon controller for this type of systems. We use a successive approximation algorithm to solve the quadratic optimal control problem subject to a bilinear PDE system, i.e, the bilinear quadratic regulator (BQR) problem. By invoking appropriate optimal conditions for the optimal control problem, a nonlinear two-point boundary value (TPBV) problem is usually obtained. Unfortunately, a closed-form solution of such nonlinear TPBV problem is difficult to be obtained in general. Inspired by the work on finite-dimensional systems in [8], we propose a successive approximation approach to solve the nonlinear TPBV problem for an infinite-dimensional system. For a linear quadratic optimal control problem, the analytical solution of the associated TPBV problem is possible. Moreover, the gain of the resulting optimal control law is independent of the initial conditions, and the optimal control is asymptotic stabilizing and robust to disturbances. As these desired properties no longer hold for the bilinear case, a receding horizon control framework is proposed to guarantee asymptotic stability for the proposed control approach. The two most important theoretical contributions of this work are the convergence proof of the successive approximation algorithm implemented to solve the BQR problem and the stability proof of the proposed receding horizon control scheme for the bilinear infinite dimensional system.

This paper is organized as follows. In Section II, we provide the functional setting and necessary technical terms that will be used in this paper. A parabolic PDE with bilinear control is introduced and formulated as an infinite-dimensional system. The quadratic optimal problem and a finite-horizon nonlinear RHC scheme to solve it are also presented in this section. We propose in Section III a successive approximation method used to numerically solve the optimal control problem, and the convergence proof of the algorithm is provided as well. We discuss feasibility and stability of the proposed nonlinear RHC scheme in Section IV. Section V illustrates the effectiveness of the proposed feedback controller in addressing the current profile control problem in tokamaks. Finally, conclusions and future work are presented in Section VI.

This work was supported by the NSF CAREER award program (ECCS-0645086).

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II. PRELIMINARIES

We define the following functional space

$$L^2(\Omega) = \left\{ f(x) \middle| \int_\Omega f^2(x)dx < \infty \right\}, \quad (1)$$

$$H^1(\Omega) = \left\{ f | f \in L^2(\Omega) \text{ and } f' \in L^2(\Omega) \right\}. \quad (2)$$

Throughout this paper, $\mathbb{R}^{m,n}$ denotes the space of $m \times n$ real matrices. For a matrix $A \in \mathbb{R}^{m,n}$, A^T represents the transpose. In general, we restrict ourselves to the case of Hilbert spaces, where inner products, norms, and operator norms are denoted by $\langle \cdot, \cdot \rangle$, $|\cdot|$, and $\|\cdot\|$, respectively. In case of an operator A, the adjoint operator is denoted by A^*. The term "a.e." denotes almost everywhere. For additional information on semigroups of operators, the reader is referred to [9].

A. Bilinear Parabolic PDEs

Consider an *infinite-dimensional system* in Hilbert spaces Z and W defined by the following relationship

$$\frac{dz}{dt} = Az + Bzv = f(z(t), v(t)), \quad \text{for } t > 0 \quad (3)$$

with

$$z(0) = z_0, \text{ and } v \in U,$$

where z is the state and $z_0 \in Z$; A is the generator of a strongly continuous semigroup on Z; $B \in \mathscr{L}(Z)$ is a bounded linear operator; v is the control and U is a closed convex subset of W; $z(t)$ is strongly continuous on $[0,T]$ for all input $v \in L^2([0,T];U)$.

We define a finite horizon cost functional associated with the system (3) as

$$\begin{aligned} J(v,t,z) =& \frac{1}{2}\langle z(T), Sz(T)\rangle \\ &+ \frac{1}{2}\int_0^T (\langle z(t), Qz(t)\rangle + \langle v(t), Rv(t)\rangle)\, dt, \end{aligned} \quad (4)$$

where $S, Q \in \mathscr{L}(Z)$ and $R \in \mathscr{L}(W)$ are self-adjoint, positive and

$$\langle v(t), Rv(t)\rangle \geq a\|v\|^2, \ \forall v \in U, \text{and some } a > 0. \quad (5)$$

The optimal control problem is to find a control $v^\star \in L^2([0,T],U)$, which minimizes $J(v,t,z)$, i.e., which drives the state $z(t)$ close to the zero state in a finite-time horizon.

For the purpose of this introductory discussion, we assume that for every $z_0 \in Z$ and $v \in L^2([0,T],U)$ there exist a continuous $z(t) = z(t, z_0, v)$, which is a weak solution of (3). By defining the Hamiltonian H as

$$H(z,v,\Pi) = \frac{1}{2}\langle z, Qz\rangle + \frac{1}{2}\langle v, Rv\rangle + \langle \Pi, (Az + Bzv)\rangle, \quad (6)$$

where Π denotes the infinite-dimensional Lagrange multiplier, and by invoking the minimum principle, the optimal problem reduces to solving a nonlinear two-point boundary value (TPBV) problem [4],

$$\begin{cases} \dfrac{dz}{dt} = \dfrac{\partial H}{\partial \Pi}, & z(0) = z_0 \\ \dfrac{d\Pi}{dt} = -\dfrac{\partial H}{\partial z}, & \Pi(T) = Sz(T). \end{cases} \quad (7)$$

A closed-form solution of the nonlinear TPBV problem (7) is extremely difficult to obtain, if not impossible, even in a finite dimensional formulation. It is necessary to find approximate approaches for solving the optimal problem of the bilinear PDE systems.

B. Receding Horizon Control Formulation

Referring to system (3), we consider the problem of asymptotic stabilization of the origin, subject to the control constraints of $v \in L^2([0,T],U)$. The problem will be addressed within a receding horizon control (RHC) framework (see [10] for a review of various RHC algorithms for finite-dimensional systems) where the control v and state z at time t are conventionally obtained by solving, repeatedly, a finite horizon constrained optimal control problem of the form,

$$\begin{aligned} \min_{v \in U} J =& \frac{1}{2}\int_{T_i}^{T_{i+1}} (\langle z(t), Qz(t)\rangle + \langle v(t), Rv(t)\rangle)\, dt \\ &+ \frac{1}{2}\langle z(T_{i+1}), s(t) z(T_{i+1})\rangle, \end{aligned} \quad (8)$$

subject to

$$\frac{dz}{dt} = Az + Bzv = f(z,v), \qquad t \in [T_i, T_{i+1}], \quad (9)$$

where $s(t) \in \mathscr{L}(Z)$ denotes a self-adjoint positive operator. Let $0 = T_0 < T_1 < \ldots < T_n = T$ describe a grid of equal intervals on $[0,T]$ and let $\Delta T = T_{i+1} - T_i$, $i = 0, 1, \ldots, n-1$.

III. A SUCCESSIVE APPROXIMATION ALGORITHM AND ITS CONVERGENCE

A. An Iterative Scheme

In this section, we propose a successive approximation approach to solve the TPBV problem (7) for infinite-dimensional systems by extending the successive approximation approach for finite-dimensional systems in [11]. By expanding our problem (3) up to first-order around the previous iteration trajectories $z^{(k)}(t)$ and $u^{(k)}(t)$, the system (3) takes the form

$$\dot{z}^{(k+1)} = Az^{(k+1)} + B^{(k)} v^{(k+1)}, \quad (10)$$

where k is the iteration number and

$$B^{(k)}(t) = Bz(t)|_{z=z^{(k)}(t)}, \quad (11)$$

with initial condition $z^{(k+1)}(0) = z_0$. The cost function takes the form

$$\begin{aligned} J^{(k+1)} =& \frac{1}{2}\langle z^{(k+1)}(T_{i+1}), Sz^{(k+1)}(T_{i+1})\rangle \\ &+ \frac{1}{2}\int_{T_i}^{T_{i+1}} \langle z^{(k+1)}(t), Qz^{(k+1)}(t)\rangle \\ &+ \langle v^{(k+1)}(t), Rv^{(k+1)}(t)\rangle dt, \end{aligned} \quad (12)$$

and the Hamilton is written as

$$\begin{aligned} &H(z^{(k+1)}, v^{(k+1)}, \Pi^{(k+1)}) \\ =& \frac{1}{2}\langle z^{(k+1)}(t), Qz^{(k+1)}(t)\rangle + \frac{1}{2}\langle v^{(k+1)}(t), Rv^{(k+1)}(t)\rangle \\ &+ \langle \Pi^{(k+1)}(t), (Az^{(k+1)}(t) + B^{(k)} v^{(k+1)}(t))\rangle. \end{aligned} \quad (13)$$

For each iteration, we have an abstract linear quadratic optimal control problem defined by (10) and (12) with the approximate control law given by

$$v^{(k+1)}(t)=-R^{-1}(B^{(k)})^{*}\Pi^{(k+1)}(t). \quad (14)$$

As explained above, the optimal problem is characterized by the following TPBV problem,

$$\begin{cases} \dot{z}^{(k+1)} = Az^{(k+1)}+B^{(k)}(-R^{-1})(B^{(k)})^{*}\Pi^{(k+1)}(t) \\ \dot{\Pi}^{(k+1)} = -Qz^{(k+1)}-A^{*}\Pi^{(k+1)}, \end{cases} \quad (15)$$

along with the boundary conditions

$$z^{(k+1)}(T_i)=\bar{z}(T_i), \quad \Pi^{(k+1)}(T_{i+1})=Sz^{(k+1)}(T_{i+1}), \quad (16)$$

where $\bar{z}(T_i)$ refers to the measured value.

Let us propose the solution form

$$\Pi^{(k+1)}(t)\triangleq s^{(k+1)}(t)z^{(k+1)}(t), \quad (17)$$

where $s^{(k+1)}(t)\in\mathscr{L}(Z)$ is a linear operator. By substituting (17) into (15), we can obtain the following differential Riccati equation

$$\dot{s}^{(k+1)}=-s^{(k+1)}A-A^{*}s^{(k+1)}-Q+s^{(k+1)}B^{(k)}R^{-1}(B^{(k)})^{*}s^{(k+1)}, \quad (18)$$

with terminal condition

$$s^{(k+1)}(T_{i+1})=S.$$

Assume that for all $z(t)\in Z$ and $t\in[T_i,T_{i+1}]$, (A,B^k) is stabilizable and Q is appropriately designed, such that the Riccati equation (18) has a unique positive solution.

Then, the closed-loop system becomes

$$\dot{z}^{(k+1)}=Az^{(k+1)}-B^{(k)}R^{-1}(B^{(k)})^{*}s^{(k+1)}z^{(k+1)}, \quad (19)$$

with the initial condition $z^{(k+1)}(T_i)=\bar{z}(T_i)$.

The open-loop state trajectories $z^o(t)$ are used to evaluate (11) and start the iterations. The iterative procedure is stopped when convergence is achieved under given error tolerance. Finally, by using the convergent solution $s(t)$ of the Riccati equations (18), we obtain the following feedback control law

$$v^{\star}(t)=-R^{-1}(Bz^{\star}(t))^{*}s^{\star}(t)z^{\star}(t). \quad (20)$$

where $\star$ denotes the converged values of the iteration. The optimal trajectory $z^{\star}(t)$ driven by $v^{\star}(t)$ is

$$\dot{z}^{\star}=(A-Bz^{\star}R^{-1}(Bz^{\star}(t))^{*}s^{\star})z^{\star}. \quad (21)$$

B. Proof of Convergence for the Iterative Scheme

In the rest of this section, it remains to prove the convergence of the proposed successive approximation approach in solving the optimal control problem. Namely, we will show the following limits in appropriate functional spaces

$$\lim_{k\to\infty}\mathrm{z}^{(k)}=\mathrm{z}^{\star}, \quad \lim_{k\to\infty}s^{(k)}=s^{\star}. \quad (22)$$

The associated spaces are two Banach spaces (see, e.g., [12])

$$\mathfrak{B}_1=\mathfrak{B}_2=C([T_i,T_{i+1}],Z), \quad (23)$$

with norms $\|\mathrm{z}\|_{\mathfrak{B}_1}=\sup_{\tau\in[T_i,T_{i+1}]}\|\mathrm{z}(\tau)\|$, for any $\mathrm{z}\in\mathfrak{B}_1$ and $\|s\|_{\mathfrak{B}_2}=\sup_{\tau\in[T_i,T_{i+1}]}\|s(\tau)\|$, for any $s\in\mathfrak{B}_2$, where $\|\mathrm{z}\|=\sqrt{\langle z,z\rangle}$ and $\|s\|=\sqrt{\langle s,s\rangle}$.

Remark 1: To show (22), we only need to show that both $\left\{\mathrm{z}^{(k)}\right\}$ and $\left\{s^{(k)}\right\}$ are Cauchy sequences. Thus, the convergence follows due to the completeness of the Banach spaces. The convergence proof is based on the contraction mapping theorem for Banach spaces [12], which is motivated by the convergence proof in [13].

Based on (18) and (19), we obtain differential equations for the differences $\mathrm{z}^{(k+1)}-\mathrm{z}^{(k)}$ and $s^{(k+1)}-s^{(k)}$, i.e.,

$$\begin{aligned} &\frac{d}{dt}\left[\mathrm{z}^{(k+1)}-\mathrm{z}^{(k)}\right]=\mathfrak{A}^{(k)}\mathrm{z}^{(k+1)}-\mathfrak{A}^{(k-1)}\mathrm{z}^{(k)} \\ &=\mathfrak{A}^{(k)}\left(\mathrm{z}^{(k+1)}-\mathrm{z}^{(k)}\right)+\left(\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right)\mathrm{z}^{(k)}, \end{aligned} \quad (24)$$

$$\begin{aligned} &\frac{d}{dt}\left[s^{(k+1)}-s^{(k)}\right]+\left[s^{(k+1)}-s^{(k)}\right]\mathfrak{A}^{(k)} \\ &+\mathfrak{A}^{(k-1)*}\left[s^{(k+1)}-s^{(k)}\right]+s^{(k)}\left[\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right] \\ &+\left[\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right]^{*}s^{(k+1)}+\mathfrak{Q}^{(k)}-\mathfrak{Q}^{(k-1)}=0, \end{aligned} \quad (25)$$

where

$$\mathfrak{A}^{(k)}=A-B^{(k)}R^{-1}\left(B^{(k)}\right)^{*}s^{(k+1)}, \quad (26)$$

$$\mathfrak{Q}^{(k)}=Q+s^{(k+1)}B^{(k)}R^{-1}\left(B^{(k)}\right)^{*}s^{(k+1)}. \quad (27)$$

In order to express the solutions of (24) and (25), we introduce the transition operator $\Phi^{(k)}(t,T_i)$ which solves

$$\dot{\Phi}^{(k+1)}(t,T_i) = \mathfrak{A}^{(k)}(t)\Phi^{(k+1)}(t,T_i), \quad (28)$$

$$\Phi^{(k+1)}(T_i,T_i) = I. \quad (29)$$

In the subsequent proof we will use some of the following properties of the transition operator $\Phi(\cdot,\cdot)$:

$$\Phi(t,\tau)\Phi(\tau,T_i)=\Phi(t,T_i), \quad \Phi^{-1}(t,\tau)=\Phi(\tau,t). \quad (30)$$

The following lemma provides solutions for (24) and (25).

Lemma 1: The solutions of (24) and (25) are

$$\begin{aligned} &\mathrm{z}^{(k+1)}-\mathrm{z}^{(k)} \\ &=\int_{T_i}^{t}\Phi^{(k+1)}(t,\tau)\left(\mathfrak{A}^{(k)}(\tau)-\mathfrak{A}^{(k-1)}(\tau)\right)\Phi^{(k)}(\tau,T_i)\mathrm{z}^0 d\tau, \end{aligned} \quad (31)$$

and

$$\begin{aligned} s^{(k+1)}-s^{(k)}=&\int_{t}^{T_{i+1}}[\Phi^{(k)}(\tau,t)]^{*} \\ &\times\left\{s^{(k)}\left[\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right]+\left[\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right]^{*}s^{(k+1)}\right. \\ &\left.+\mathfrak{Q}^{(k)}-\mathfrak{Q}^{(k-1)}\right\}\Phi^{(k+1)}(\tau,t)d\tau. \end{aligned} \quad (32)$$

Proof: The integral expression for $\mathrm{z}^{(k+1)}-\mathrm{z}^{(k)}$ can be obtained by directly integrating both sides of the linear system (24). This expression is written in terms of the transition operator $\Phi(t,T_i)$ defined in (28)–(30). We note that the initial value of the difference term $\mathrm{z}^{(k+1)}(T_i)-\mathrm{z}^{(k)}(T_i)=0$ due to (16). Therefore, only the inhomogeneous term of the solution appears in (31). Additionally, we use the transition operator to write $\mathrm{z}^{(k)}(\tau)=\Phi^{(k)}(\tau,T_i)\mathrm{z}_0$ in (31).

For the integral expression for $s^{(k+1)} - s^{(k)}$, we first use the definition (28) of the transition operator to compute the derivative in time

$$\begin{aligned}&\frac{d}{dt}\left\{[\Phi^{(k)}(t,T_i)]^*\left[s^{(k+1)}-s^{(k)}\right]\Phi^{(k+1)}(t,T_i)\right\}\\&=[\Phi^{(k)}(t,T_i)]^*[\mathfrak{A}^{(k-1)}(t)]^*\left[s^{(k+1)}-s^{(k)}\right]\Phi^{(k+1)}(t,T_i)\\&+[\Phi^{(k)}(t,T_i)]^*\left[s^{(k+1)}-s^{(k)}\right]\mathfrak{A}^{(k)}(t)\Phi^{(k+1)}(t,T_i)\\&+[\Phi^{(k)}(t,T_i)]^*\frac{d}{dt}\left[s^{(k+1)}-s^{(k)}\right]\Phi^{(k+1)}(t,T_i).\end{aligned}\tag{33}$$

Then, we use (25) to rewrite (33)

$$\begin{aligned}&\frac{d}{dt}\left\{[\Phi^{(k)}(t,T_i)]^*\left[s^{(k+1)}-s^{(k)}\right]\Phi^{(k+1)}(t,T_i)\right\}\\&=[\Phi^{(k)}(t,T_i)]^*\left\{\left[\mathfrak{A}^{(k-1)}-\mathfrak{A}^{(k)}\right]^*s^{(k+1)}+\mathfrak{Q}^{(k-1)}\right.\\&\quad\left.-\mathfrak{Q}^{(k)}+s^{(k)}\left[\mathfrak{A}^{(k-1)}-\mathfrak{A}^{(k)}\right]\right\}\Phi^{(k+1)}(t,T_i).\end{aligned}\tag{34}$$

Integrating both sides from t to T_{i+1}, we can obtain

$$\begin{aligned}&[\Phi^{(k)}(t,T_i)]^*\left[s^{(k+1)}-s^{(k)}\right]\Phi^{(k+1)}(t,T_i)\\&=\int_t^{T_{i+1}}[\Phi^{(k)}(\tau,T_i)]^*\left\{s^{(k)}\left[\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right]\right.\\&+\left[\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right]^*s^{(k+1)}\\&\left.+\mathfrak{Q}^{(k)}-\mathfrak{Q}^{(k-1)}\right\}\Phi^{(k+1)}(\tau,T_i)d\tau,\end{aligned}\tag{35}$$

where the final difference term $s^{(k+1)}(T_{i+1}) - s^{(k)}(T_{i+1})$ vanishes due to the terminal condition $s^{(k+1)}(T_{i+1}) = S$ in (18). In order to cancel $[\Phi^{(k)}(t,T_i)]^*$ and $\Phi^{(k+1)}(t,T_i)$ in (35), we multiply both sides of the equation (35) with $[\Phi^{(k)}(T_i,t)]^*$ (from the left) and $\Phi^{(k+1)}(T_i,t)$ (from the right) respectively, and use (30) to obtain the integral expression for $s^{(k+1)} - s^{(k)}$. ■

Theorem 2: There exists an appropriate control weight operator R, such that the sequences $\{z^{(k)}(t)\}$ and $\{s^{(k)}(t)\}$ generated by (18) and (19) are convergent.

Proof: Taking the $\|\cdot\|_{\mathfrak{B}}$–norm of $z^{(k+1)} - z^{(k)}$ and $s^{(k)} - s^{(k-1)}$ derived in Lemma 1, we have

$$\left\|z^{(k+1)}-z^{(k)}\right\|_{\mathfrak{B}_1}\le\mu_1\left\|\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right\|_{\mathfrak{B}_2}\tag{36}$$

$$\begin{aligned}\left\|s^{(k+1)}-s^{(k)}\right\|_{\mathfrak{B}_2}&\le\mu_2\left\|\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right\|_{\mathfrak{B}_2}\\&+\mu_3\left\|\mathfrak{Q}^{(k)}-\mathfrak{Q}^{(k-1)}\right\|_{\mathfrak{B}_2}\end{aligned}\tag{37}$$

where

$$\begin{aligned}\mu_1&=\max_{T_i\le\tau\le t\le T_{i+1}}\left\|\Phi^{(k+1)}(t,\tau)\right\|\left\|\Phi^{(k)}(\tau,T_i)\right\|\|z_0\|,\\\mu_2&=\max_{T_i\le t\le\tau\le T_{i+1}}\left\|\Phi^{(k)}(\tau,t)\right\|\left(\|s^{(k+1)}\|+\|s^{(k)}\|\right)\\&\quad\times\left\|\Phi^{(k+1)}(\tau,t)\right\|,\\\mu_3&=\max_{T_i\le t\le\tau\le T_{i+1}}\left\|\Phi^{(k)}(\tau,t)\right\|\left\|\Phi^{(k+1)}(\tau,t)\right\|.\end{aligned}$$

By noting the definitions (26) and (27), and by defining $\mathscr{S}^{(k)} \triangleq B^{(k)}R^{-1}B^{(k)^*}$, we obtain the following norm bounds,

$$\begin{aligned}\left\|\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right\|_{\mathfrak{B}_2}&=\left\|-\mathscr{S}^{(k-1)}s^{(k)}+\mathscr{S}^{(k)}s^{(k+1)}\right\|_{\mathfrak{B}_2}\\&\le\left\|\left(\mathscr{S}^{(k)}-\mathscr{S}^{(k-1)}\right)s^{(k+1)}\right\|_{\mathfrak{B}_2}\\&\quad+\left\|\mathscr{S}^{(k-1)}\left(s^{(k+1)}-s^{(k)}\right)\right\|_{\mathfrak{B}_2}\end{aligned}\tag{38}$$

$$\begin{aligned}\left\|\mathfrak{Q}^{(k)}-\mathfrak{Q}^{(k-1)}\right\|_{\mathfrak{B}_2}&\le\left\|s^{(k+1)}-s^{(k)}\right\|_{\mathfrak{B}_2}\left\|\mathscr{S}^{(k)}s^{(k+1)}\right\|_{\mathfrak{B}_2}\\&+\left\|s^{(k)}\right\|_{\mathfrak{B}_2}\left\|\mathscr{S}^{(k)}-\mathscr{S}^{(k-1)}\right\|_{\mathfrak{B}_2}\left\|s^{(k)}\right\|_{\mathfrak{B}_2}\\&+\left\|s^{(k)}\mathscr{S}^{(k)}\right\|_{\mathfrak{B}_2}\left\|s^{(k+1)}-s^{(k)}\right\|_{\mathfrak{B}_2}.\end{aligned}\tag{39}$$

Now we connect the terms in (38) and (39) with the factors $\left\|z^{(k+1)}-z^{(k)}\right\|_{\mathfrak{B}_2}$ and $\left\|s^{(k+1)}-s^{(k)}\right\|_{\mathfrak{B}_2}$ to obtain

$$\begin{aligned}\left\|\mathscr{S}^{(k)}-\mathscr{S}^{(k-1)}\right\|_{\mathfrak{B}_2}&\le\left\|B^{(k)}-B^{(k-1)}\right\|_{\mathfrak{B}_2}\left\|R^{-1}B^{(k)^*}\right\|_{\mathfrak{B}_2}\\&+\left\|B^{(k-1)}R^{-1}\right\|_{\mathfrak{B}_2}\left\|B^{(k)^*}-B^{(k-1)^*}\right\|_{\mathfrak{B}_2}\\&\le\frac{\left(\left\|B^{(k)^*}\right\|_{\mathfrak{B}_2}+\left\|B^{(k-1)}\right\|_{\mathfrak{B}_2}\right)\|K\|_{\mathfrak{B}_2}}{\|R\|}\\&\quad\times\left\|z^{(k)}-z^{(k-1)}\right\|_{\mathfrak{B}_2}.\end{aligned}\tag{40}$$

Using the norm bound estimates in (36)–(40), we obtain

$$\begin{aligned}&\left\|z^{(k+1)}-z^{(k)}\right\|_{\mathfrak{B}_1}\\&\le\nu_1\left\|s^{(k+1)}-s^{(k)}\right\|_{\mathfrak{B}_2}+\nu_2\left\|z^{(k)}-z^{(k-1)}\right\|_{\mathfrak{B}_2}\end{aligned}\tag{41}$$

where μ_1 and μ_2 are defined by

$$\nu_1=\mu_1\left\|\mathscr{S}^{(k-1)}\right\|_{B_2},\tag{42}$$

$$\nu_2=\mu_1\|s^{(k+1)}\|_{B_2}\frac{\left(\left\|B^{(k)^*}\right\|_{B_2}+\left\|B^{(k-1)}\right\|_{\mathfrak{B}_2}\right)\|K\|_{\mathfrak{B}_2}}{\|R\|},\tag{43}$$

and

$$\begin{aligned}&\left\|s^{(k+1)}-s^{(k)}\right\|_{\mathfrak{B}_2}\\&\le\nu_3\left\|s^{(k+1)}-s^{(k)}\right\|_{\mathfrak{B}_2}+\nu_4\left\|z^{(k)}-z^{(k-1)}\right\|_{\mathfrak{B}_2}\end{aligned}\tag{44}$$

where

$$\begin{aligned}\nu_3=\mu_2\left\|\mathscr{S}^{(k-1)}\right\|_{\mathfrak{B}_2}\mu_3\left\|\mathscr{S}^{(k)}\right\|_{\mathfrak{B}_2}\\\times(\left\|s^{(k)}\right\|_{\mathfrak{B}_2}+\left\|s^{(k+1)}\right\|_{\mathfrak{B}_2}),\end{aligned}\tag{45}$$

$$\nu_4=\frac{\mu_2\nu_2}{\mu_1}+\frac{\mu_3\nu_2}{\mu_1}\frac{\left\|s^{(k)}\right\|^2_{\mathfrak{B}_2}}{\left\|s^{(k+1)}\right\|_{\mathfrak{B}_2}}.\tag{46}$$

We note that (44) can be solved with respect to $\left\| s^{(k+1)} - s^{(k)} \right\|_{\mathfrak{B}_2}$, i.e.,

$$\left\| s^{(k+1)} - s^{(k)} \right\|_{\mathfrak{B}_2} \leq \frac{\nu_4}{1-\nu_3} \left\| \mathrm{z}^{(k)} - \mathrm{z}^{(k-1)} \right\|_{\mathfrak{B}_2}. \quad (47)$$

By substituting (47) into (41), we obtain

$$\left\| \mathrm{z}^{(k+1)} - \mathrm{z}^{(k)} \right\|_{\mathfrak{B}_1} \leq \frac{\nu_2 + \nu_4(\nu_1 - \nu_2)}{1-\nu_4} \left\| \mathrm{z}^{(k)} - \mathrm{z}^{(k-1)} \right\|_{\mathfrak{B}_2}.$$

At this point, it is important to mention that because of the multiplicative influence of R^{-1} in equations (43) and (46) for ν_2 and ν_4, respectively, if $\|R\|$ is large enough, we can make sure that the coefficients involved are less than one, i.e.,

$$\max\left\{ \left| \frac{\nu_4}{1-\nu_3} \right|, \left| \frac{\nu_2 + \nu_4(\nu_1 - \nu_2)}{1-\nu_4} \right| \right\} < 1. \quad (48)$$

Thus, we can conclude that both $\{s^{(k)}\}$ and $\{\mathrm{z}^{(k)}\}$ are Cauchy sequences in the associated Banach spaces, i.e., $\left\| s^{(k+1)} - s^{(k)} \right\|_{\mathfrak{B}_2} \to 0$, $\left\| \mathrm{z}^{(k+1)} - \mathrm{z}^{(k)} \right\|_{\mathfrak{B}_1} \to 0$. Due to the completeness of the Banach space, any Cauchy sequence in such a complete space is convergent, thus

$$\lim_{k\to\infty} s^{(k)}(t) = s^{\star}(t), \quad (49)$$

$$\lim_{k\to\infty} \mathrm{z}^{(k)}(t) = \mathrm{z}^{\star}(t). \quad (50)$$

■

IV. ASYMPTOTIC STABILITY PROPERTY

The stability properties of the proposed receding horizon control approach must be theoretically justified. Namely, assuming that $z = 0$ is an equilibrium for (3) with $v = 0$, we need to prove that such equilibrium can be stabilized by means of an optimal control formulation with $T \to \infty$.

According to the principle of RHC, the open-loop optimal control problem given by equation (8) will be solved repeatedly, updated with new measurements $\bar{z}(T_i)$, $(i = 0,1,2,...,n-1)$. The closed-loop control $\bar{v}(\cdot)$ is defined by

$$\bar{v}(\tau) = v^{\star}(\tau; \bar{z}(T_i), [T_i, T_{i+1}]), \qquad \tau \in [T_i, T_{i+1}], \quad (51)$$

where $v^{\star}(\cdot)$ in (20) is the solution of the open-loop optimal problem (8). In this section, we study the stability properties of the closed-loop system

$$\dot{z}(t) = f(z(t), \bar{v}(t)). \quad (52)$$

Due to the repeated solution of the optimal problem described by equations (8) and (9), feasibility is required at each time $t \geq 0$. Here, feasibility of the optimal problem means that there exists at least one (not necessarily optimal) control input trajectory $v(\cdot) : [T_i, T_{i+1}] \to U$, such that the value of the cost functional (8) is bounded.

Lemma 3: For the nominal system (9) with no disturbance, the feasibility of the open-loop control problem (8) subject to equations (9) at time $t = T_i$ $(T_i \geq 0)$ implies its feasibility for all $t > T_i$.

The proof for Lemma 3 can be achieved by following an analogous proof for a finite dimensional systems in our previous work [14].

Theorem 4: Suppose that the open-loop control problem (4) subject to (3) is feasible at $t = 0$. Then in the absence of disturbances, the closed-loop system with the model predictive control (20) is nominally asymptotically stable.

Proof: According to Lemma 3, feasibility of the open-loop control problem at each time $t > 0$ is guaranteed by the assumption in the theorem.

For $\bar{z}(t) = 0$, the optimal solution to the optimization problem (8) is $v^{\star}(\cdot; \bar{z}(t), [t, T_{i+1}]) \to 0$, i.e., $\bar{v}^{\star}(\tau) = 0$, $\forall\, \tau \in [t, t+\Delta T]$. Due to $f(0,0) = 0$ in (3), then $\bar{z}(t) = 0$ is an equilibrium of the closed-loop system (52).

The key point of this proof is that in the absence of disturbance, driven by control $\bar{v}(t)$, the closed-loop states $z(t)$ will always follow an open-loop optimal trajectory $z^{\star}(t)$ in (21) controlled by the corresponding $v^{\star}(t)$ in (20).

Now we define a function $G(z)$ for the closed-loop system (52) as follows:

$$G(z(t)) = \langle z(t), s^{\star}(t) z(t) \rangle, \quad (53)$$

where for any given $\bar{z}(0) \in Z$, $s^{\star}(t)$ is the solution of the differential Riccati equation (18) after the successive approximation algorithm converges. Then, $G(z)$ has the following properties:

(1) $G(0) = 0$ and $G(z) > 0$ for $z \neq 0$,

(2) along the trajectory of the closed-loop system starting from $z_0 \in Z$, using equations (11), (18) and (19)

$$\begin{aligned} \dot{G}(z) =& \langle \dot{z}, sz \rangle + \langle z, s\dot{z} \rangle + \langle z, \dot{s}z \rangle \\ =& \langle z, (sA + A^{*}s - 2sBzR^{-1}(Bz)^{*}s)z \rangle + \langle z, \dot{s}z \rangle \\ =& -\langle z, (Q + sBzR^{-1}(Bz)^{*}s)z \rangle. \end{aligned} \quad (54)$$

Since Q, R are positive, we obtain,

$$\dot{G}(z) \leq -\langle z, Qz \rangle. \quad (55)$$

Let $s_m = \sup_{t\in[0,T]}(\|s^{\star}(t)\|)$ and $w = \|\frac{Q}{s_m}\|$. Because we can design large enough Q in (18) to make $s^{\star}(t)$ monotonously decreasing and S is positive, $s^{\star}(t)$ is positive for all $t \in [0,T]$ and therefore, $s_m > 0$ and $w > 0$. Then, since

$$\langle z, Qz \rangle \geq \langle z, s^{\star}wz \rangle = w\langle z, s^{\star}z \rangle = wG(z), \text{ for a.e. } t \in [0,T], \quad (56)$$

we can write (55) as

$$\dot{G}(z) \leq -wG(z), \text{ for a.e. } t \in [0,T]. \quad (57)$$

Multiplying by $\exp(wt)$ and integrating on $[0,T]$ implies that

$$G(T) \leq \exp(-wT)G(0). \quad (58)$$

As $T \to \infty$, $G(T) \to 0$ and then $z(T) \to 0$. Therefore, the closed-loop system (52) is asymptotically stable. ■

V. SIMULATION STUDY

In this section, the proposed approach is applied to the control of the current profile in tokamak plasmas and its effectiveness is demonstrated in simulations for a disturbance rejection problem.

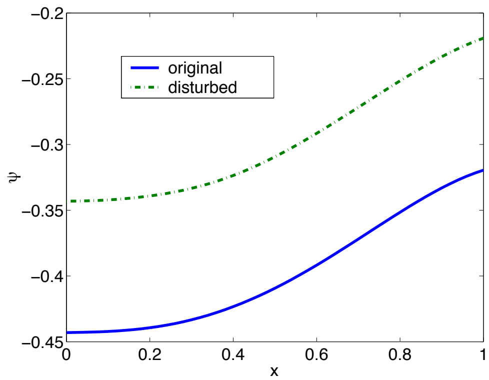


Fig. 1. Comparison of initial ψ profiles.

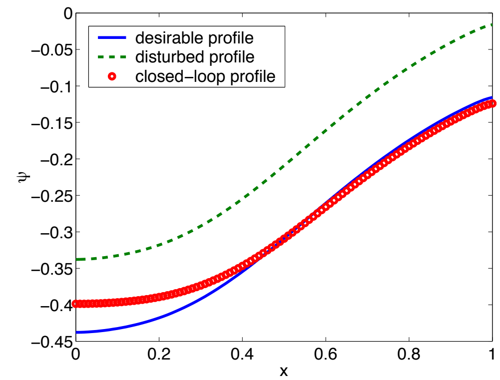


Fig. 2. Final time ψ matching comparison.

A. Current Profile Evolution Model

A key goal in the control of a magnetic fusion reactor is to maintain current profiles that are compatible with a high fraction of the self-generated non-inductive current as well as with magnetohydrodynamic (MHD) stability at high plasma pressure. This enables high fusion gain and noninductive sustainment of the plasma current for steady-state operation. The dynamics of the poloidal flux ψ is governed in normalized cylindrical coordinates by a nonlinear parabolic partial differential equation (PDE) usually referred to as the magnetic diffusion equation, where the spatial coordinate corresponds to the minor radius of the torus [3],

$$\frac{\partial \psi}{\partial t} = h_1(x)u_1(t)\frac{\partial}{\partial x}\left(h_4(x)\frac{\partial \psi}{\partial x}\right) + h_2(x)u_2(t), \quad (59)$$

with boundary conditions

$$\left.\frac{\partial \psi}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial \psi}{\partial x}\right|_{x=1} = k_3 u_3(t), \quad (60)$$

and where

$$u_1(t) = \left(\frac{n_{ave}(t)}{I(t)\sqrt{P_{tot}}}\right)^{3/2}, \; u_2(t) = \frac{\sqrt{P_{tot}(t)}}{I(t)}, \; u_3(t) = I(t).$$

We consider the line-averaged density $n_{ave}(t)$, the plasma current $I(t)$, and the total auxiliary power $P_{tot}(t)$ as the physical actuators of the system.

The control objective is to drive $\psi(x,t)$ from any arbitrary initial profile to a prescribed target or desirable profile profile $\psi^{des}(x)$ at some time T. In this work we assume that such optimal control problem has been solved off-line for a nominal initial profile, and we design a feedback controller based on RHC to track the off-line optimal trajectory when perturbations are present in the initial profile.

B. Tracking Control Problem Description

To simplify the expression for the system model (59), we define a bounded linear operator $\mathscr{A}$ in an appropriate space as

$$\mathscr{A}\phi = h_1(x)u_1(t)\frac{\partial}{\partial x}\left(h_4(x)\frac{\partial \phi}{\partial x}\right), \qquad (0 \le x \le 1). \quad (61)$$

We let $u^o(t) = [u_1^o \; u_2^o \; u_3^o]^T$ be a set of open-loop control trajectories, which are computed off-line, and $\psi^o(t)$ be the open-loop state trajectory associated with the open-loop control $u^o(t)$, with a nominal initial state ψ_0^o. The open-loop state trajectory satisfies

$$\dot{\psi}^o = \mathscr{A}\psi^o u_1^o + h_2 u_2^o. \quad (62)$$

with initial condition $\psi^o(t=0) = \psi_0^o$.

Let us define

$$z(t) = \psi(t) - \psi^o(t), \qquad v(t) = u(t) - u^o(t), \quad (63)$$

where $u(t) = [u_1 \; u_2 \; u_3^o]^T$ is the overall control input and $v(t) = [v_1 \; v_2 \; 0]^T$ is the to-be-designed closed-loop control, which is appended to the open-loop control $u^o(t)$. It is worth noting that the boundary control $u_3(t)$ is not considered as a control input in this optimal tracking control problem. Then, we can write

$$\frac{d\psi^o}{dt} + \frac{dz}{dt} = \mathscr{A}(\psi^o + z)(v_1 + u_1^o) + h_2(v_2 + u_2^o). \quad (64)$$

By substituting (62) into (64), we obtain

$$\frac{dz}{dt} = u_1^o \mathscr{A}z + \mathscr{A}z v_1 + \mathscr{A}\psi_o v_1 + h_2 v_2. \quad (65)$$

Defining an linear operators A in its appropriate space as

$$Az = u_1^o \mathscr{A}z, \quad (66)$$

and the input operator B as

$$Bz = [\mathscr{A}\psi_o + \mathscr{A}z, \; h_2, \; 0], \quad (67)$$

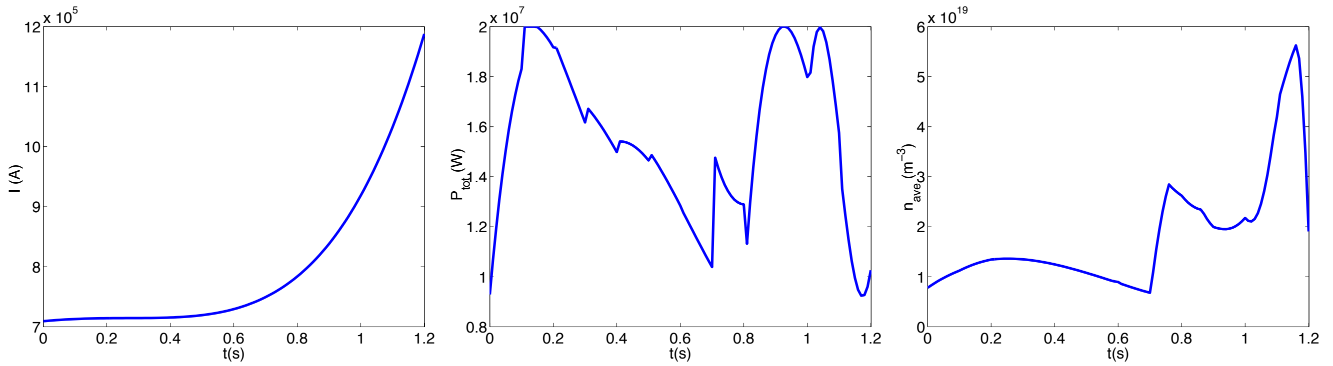
equation (65) can be written as in (3), i.e.,

$$\frac{dz}{dt} = Az + Bzv = f(z(t), v(t)). \quad (68)$$

We state the optimal tracking control problem for the state system (68) as

$$\min_{v \in U} J = \frac{1}{2}\int_0^T \left(\langle z(t), Qz(t)\rangle + v^T(t)Rv(t)\right) dt + \langle z(T), Sz(T)\rangle, \quad (69)$$

where Q, R and S are positive operators.



(a) (b) (c)

Fig. 3. RHC-based optimal tracking control trajectories: (a) $I(t)$, (b) P_{tot}, (c) $n_{ave}(t)$.

C. Simulations

In each open-loop optimal control problem of the nonlinear RHC scheme, we choose $Q = 100$, $S = 10$ and $R = diag\left\{\frac{200}{max(u_1^o)}, \frac{2}{max(u_2^o)}, 1\right\}$, for the cost functional (69), where $max(u_i^o)$ stands for the maximum value of the open-loop control $u_i^o(t)$. We use the proposed successive approximation scheme to compute the optimal control. After several iterations, the solution of the differential Riccati equation converges over each time interval, and the controller is implemented according to (20). In order to test the finite-horizon nonlinear RHC scheme, we use $\Delta T = 0.1s$ as the measurement sampling time. Each of these intervals is discretized in steps of $0.01s$ to solve the Riccati equation (18).

We consider a disturbed initial profile ψ, as shown in Fig. 1, and compare the performances of both open-loop and closed-loop controllers in the presence of this disturbance. Fig. 2 compares the final-time profiles $\psi(x,T)$, for $T = 1.2s$, obtained with both the open-loop and the closed-loop controllers, and the desired target profile $\psi^d(x)$. Both final-time profiles are obtained by considering the disturbed initial profile in Fig. 1. In the case of the open-loop controller, the control input trajectories computed in [15] for the nominal initial profile also shown in Fig. 1 are used. In the case of the closed-loop controller, the control input trajectories are shown in Fig. 3. It is possible to note from Fig. 2 that the closed-loop controller can reduce the matching error caused by the disturbances. It is also possible to note that the matching by the closed-loop controller is not perfect. However, this does not imply a limitation of the closed-loop controller but a consequence of the imposed constraints for the actuators (the R is selected to keep the actuator trajectories within physical ranges). If the actuator constraints could be reduced, the control effect would be more observable.

VI. CONCLUSIONS AND FUTURE WORKS

In this paper, we consider the design of a closed-loop optimal control law for a bilinear parabolic PDE system. A nonlinear RHC scheme using a convergent successive approximation approach to solve the finite-horizon optimal control problem is proposed. The stability analysis for the proposed scheme is also presented. A simulation study is carried out for the magnetic flux control problem in tokamak plasmas, showing that the proposed controller can overcome to some extent perturbations in the initial conditions. An optimal control design for bilinear PDE systems with state and control constraints is part of our future work.

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