POD-based Reduced Order Optimal Control of Parabolic PDE Systems via Diffusivity-Interior-Boundary Actuation

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***Abstract*— We present a framework to solve an optimal control problem for parabolic partial differential equations (PDEs) with diffusivity-interior-boundary actuators. The proposed approach is based on reduced order modeling (ROM) and successive optimal control computation. First we simulate the parabolic PDE system with given inputs to generate data ensembles, from which we then extract the most energetic modes to obtain a reduced order model based on the proper orthogonal decomposition (POD) method and Galerkin projection. The obtained reduced order model corresponds to a bilinear system. By solving the optimal control problem of the bilinear system successively, we update the given initial optimal inputs iteratively until the convergence is obtained. The simulation results demonstrate the effectiveness of the proposed method.**

I. INTRODUCTION

Many dissipative physical systems can be modeled by parabolic equations and the research concerning the associated control problems has a long history in the field of distributed parameter system (DPS) theory (e.g., [1]). Physical actuation can appear in parabolic PDEs in three different ways: source terms (interior control), boundary conditions (boundary control) and diffusivity coefficient (diffusivity control). Topics concerning interior and boundary control have been studied extensively (e.g., [2], [3] and references therein). However, control aspects of PDEs via diffusivity actuator have been seldom discussed (e.g., [4]). In this paper we consider an optimal control problem for a parabolic system with the three types of actuation mechanisms. This problem arises in the current profile control of magnetically confined fusion plasmas [5], where three physical actuators (plasma total current, average density and total power) are used to steer the plasma to achieve a desired profile in a designated time period.

The design of optimal control strategies, particularly in closed-loop, for an infinite dimensional system is often numerically unfeasible. In this case, reduced order modeling techniques may become crucial. In this paper, we use the POD method to obtain a low dimensional dynamical system (LDDS) for the parabolic PDEs. The POD method is an efficient ROM technique used to obtain LDDS's from data ensembles which arise in numerical simulation or experimental observation. The POD method has been widely used and proved successful to discover coherent structures from complex physical processes (see, e.g., [6], [7]). The obtained reduced order system in this work is a bilinear system. For the numerical solution of the optimal control, we propose an iterative method which is based on the Picard approximation [8].

This paper is organized as follows. The optimal control problem of a parabolic system is presented in Section II. In Section III, we discuss the POD method to obtain reduced order models. In Section IV, the Galerkin projection method is discussed based on a test function set composed by dominant POD modes. In Section V, we propose an iterative convergent method based on the Picard approximation to compute the optimal controls. A simulation study is presented in Section VI. Section VII closes the paper by stating the conclusions.

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II. PROBLEM STATEMENT

We consider a 1D parabolic system over

$$\Omega = \{(x,t) : 0 \le x \le L, t_0 \le t \le t_f\},$$

which is governed by

$$\begin{aligned} &\frac{\partial w}{\partial t} = \zeta(x)\left(1 + u_D(t)\right)\frac{\partial^2 w}{\partial x^2} + \xi(x) u_I(t) \\ &w(0,t) = u_B^{(0)}(t),\, w(L,t) = u_B^{(L)}(t) \\ &w(x,0) = \varphi(x) \end{aligned} \tag{1}$$

where $w(x,t)$ represents the system state, $u_D(t)$ and $u_I(t)$ the diffusivity and interior controls, respectively, $u_B^{(0)}(t)$ and $u_B^{(L)}(t)$ the boundary controls, $\varphi(x)$ the initial profile, $\zeta(x)$ and $\xi(x)$ differentiable spatial functions. For the sake of compatibility, it is necessary to assume $u_B^{(0)}(0) = \varphi(0), u_B^{(L)}(0) = \varphi(L)$. Generally, specific difficulties involved in Dirichlet control problems result from the fact that they are not of variational type (e.g., [1]). Therefore, we homogenize the boundary conditions by introducing the transformation $z(x,t) = w(x,t) - h(x,t)$, where

$$h(x,t) = \frac{x}{L} u_B^{(L)}(t) + \frac{L-x}{L} u_B^{(0)}(t). \tag{2}$$

Then, system (1) can be rewritten as

$$\begin{aligned} &\frac{\partial z}{\partial t} = \zeta(x)(1+u_D)\frac{\partial^2 z}{\partial x^2} + \xi(x) u_I(t) - \frac{\partial h(x,t)}{\partial t}, \\ &z(0,t) = 0,\, z(L,t) = 0,\, z(x,0) = \varphi(x) - h(x,0). \end{aligned} \tag{3}$$

Remark 1: We assume the existence of first order derivatives for the boundary control functions $u_B^{(0)}(t)$ and $u_B^{(L)}(t)$, which are necessary for the computation of $\frac{\partial h}{\partial t}$ in (3).

We state an optimal control problem for the parabolic system (3) with the following cost functional

$$\begin{aligned}\min_{\mathrm{u}} J = &\frac{1}{2}\int_0^L \langle z_{t_f}(x), \mathcal{S} z_{t_f}(x)\rangle dx \\ &+ \frac{1}{2}\int_\Omega \langle z, \mathcal{Q} z\rangle dx dt + \frac{1}{2}\int_{t_0}^{t_f} \langle \mathcal{R}\mathrm{u}, u\rangle dt,\end{aligned} \tag{4}$$

where $\mathcal{S}$ and $\mathcal{Q}$ are weight operators, $\mathcal{R}$ the control weight matrix, $\mathrm{u} = (u_D, u_I, u_B^{(0)}, u_B^{(L)})^T$ the control vector, $\langle\cdot,\cdot\rangle$ the inner product in appropriately defined functional spaces, and $z_{t_f}(x) = z(x, t_f)$ is the state evaluated at $t = t_f$.

III. POD Reduced Order Modeling

We first solve the parabolic PDE system by using a finite difference scheme [9] on the grid $\Omega_{ij} = (x_i, t_j)$, where i, j are integers with $1 \le i \le m; 1 \le j \le n$. The set $\mathcal{V} = \mathrm{span}\{z_1, \cdots, z_n\} \subset \mathbb{R}^m$ refers to a data ensemble consisting of the snapshots $\{z_j\}_{j=1}^n$ obtained from the simulation. Let $\{\psi_k\}_{k=1}^d$ be the orthonormal basis of the data ensemble $\mathcal{V}$, where $d = \dim\mathcal{V} \le m$. We then project each of the snapshots onto the basis ψ_k,

$$z_j = \sum_{k=1}^d (z_j, \psi_k)\psi_k, \quad j = 1, \cdots, n, \tag{5}$$

where $(\cdot,\cdot)$ denotes the inner product of the space $L^2([0, L])$. The goal of the POD method is to find an orthonormal basis such that for some predefined $1 \le l \le d$ the following average index is minimized

$$\begin{aligned}&\min_{\{\psi_k\}_{k=1}^l} \frac{1}{n}\sum_{j=1}^n \left\| z_j - \sum_{k=1}^l (z_j, \psi_k)\psi_k \right\|^2, \\ &\text{subject to} \quad (\psi_i, \psi_j) = \delta_{ij}, 1 \le i \le l, 1 \le j \le i, \\ &\text{where} \quad \|z\| = \sqrt{z^T z}.\end{aligned} \tag{6}$$

The solution of (6) can be found in the literature, e.g., [7], [10]. Defining the correlation matrix $K \in \mathbb{R}^{n\times n}$ as $K_{ij} = \frac{1}{n}(z_j, z_i)$, for $i, j = 1, \ldots, n$, it follows the following singular value decomposition result [10]:

Theorem 1: Let $\lambda_1 > \ldots > \lambda_l > \ldots > \lambda_d > 0$ denote the positive eigenvalues of the correlation matrix K and $v_1, \ldots, v_l, \ldots, v_d$ the associated eigenvectors, where $d = \mathrm{rank}(K)$. Then, the POD basis functions take the form of

$$\psi_k = \frac{1}{\sqrt{\lambda_k}}\sum_{j=1}^n (v_k)_j z_j = \frac{1}{\sqrt{\lambda_k}} Z v_k, \; (k = 1, \ldots, d), \tag{7}$$

where $(v_k)_j$ is the j-th component of the eigenvector v_k and $Z = (z_1, \cdots, z_n)$ is the collection of all the snapshots. Moreover, the error (energy ratio) associated with the approximation with the first l POD modes is

$$\varepsilon_l = \frac{1}{n}\sum_{j=1}^n \left\| z_j - \sum_{k=1}^l (z_j^T\psi_k)\psi_k \right\|^2 = \sum_{k=l+1}^d \lambda_k. \tag{8}$$

IV. POD/Galerkin Method

We introduce in this section the POD/Galerkin projection method, which is used to obtain a low dimensional dynamical system approximation of the original parabolic PDE system. We first define the weak solution of the homogenized parabolic system (3), which will be used for the POD/Galerkin projection.

Definition 1 (Weak Solution): Denoting by

$$V = H_0^1(\Omega) = \left\{ v \,\middle|\, v, \frac{\partial v}{\partial x} \in L^2(\Omega), \text{ and } v|_{\partial\Omega} = 0 \right\}$$

the test function space, then the weak solution $z(x, t)$ of the transformed system (3) satisfies

$$\begin{aligned}&\int_0^L \frac{\partial z}{\partial t} v dx + \int_0^L (1 + u_D(t))\frac{\partial z}{\partial x}\frac{\partial(\zeta v)}{\partial x} dx \\ &= \int_0^L \left(\xi(x)u_I(t) - \frac{\partial h(x,t)}{\partial t}\right) v dx,\end{aligned} \tag{9}$$

where $z, v \in V$. This expression is obtained by multiplying both sides of (3) by a test function v, and by integrating by parts.

Similar to (5), we use only $l (\le d)$ modes to implement the expansion of the transformed variable $z(x, t)$, i.e., $z(x, t) \simeq \sum_{k=1}^l \alpha_k(t)\psi_k(x)$, and substitute this expression for $z(x, t)$ and $v = \psi_j$ $(1 \le j \le l)$ into the weak form (9). Then, we can obtain the following finite dimensional system $\left(D \triangleq \frac{\partial}{\partial x}\right)$:

$$\begin{aligned}&\sum_{k=1}^l (\psi_k, \psi_j)\frac{d\alpha_k}{dt} + (1 + u_D)\sum_{k=1}^l \left(D\psi_k, D(\zeta\psi_j)\right)\alpha_k \\ &= u_I(t)(\xi, \psi_j) - \dot{u}_B^{(L)}\frac{(x, \psi_j)}{L} - \dot{u}_B^{(0)}\frac{(L - x, \psi_j)}{L}.\end{aligned} \tag{10}$$

Using the following notations

$$M_{jk} = (\psi_j, \psi_k) = \int_\Omega \psi_j(x)\psi_k(x) dx = \delta_{jk} \tag{11}$$

$$K_{jk} = -\left(D(\zeta\psi_j), D\psi_k\right) = -\int_\Omega \frac{\partial(\zeta\psi_j)}{\partial x}\frac{\partial\psi_k}{\partial x} dx \tag{12}$$

$$F_j = (\xi, \psi_j) = \int_\Omega \xi(x)\psi_j(x) dx \tag{13}$$

$$G_j = -\frac{1}{L}(L - x, \psi_j) = -\frac{1}{L}\int_\Omega (L - x)\psi_j(x) dx \tag{14}$$

$$H_j = -\frac{1}{L}(x, \psi_j) = -\frac{1}{L}\int_\Omega x\psi_j(x) dx, \tag{15}$$

we obtain a matrix representation

$$\frac{d\mathrm{y}}{dt} = K\mathrm{y} + K\mathrm{y}u_1(t) + Fu_2(t) + Gu_3(t) + Hu_4(t) \tag{16}$$

where $\mathrm{y}(t) = (\alpha_1(t), \ldots, \alpha_l(t))^T \in \mathbb{R}^l$, $M, K \in \mathbb{R}^{l\times l}$, $F, G, H \in \mathbb{R}^l$, and $\mathrm{u} = (u_1, u_2, u_3, u_4)^T = (u_D, u_I, u_B^{(0)}, u_B^{(L)})^T \in \mathbb{R}^4$. The vector $\mathrm{y}(t)$ is the finite dimensional approximation, with respect to the obtained POD modes, of the variable $z(x, t)$ in (3). The initial values are given by $\alpha_j(0) = (z(\cdot, 0), \psi_j)_{L^2}, j = 1, 2, \cdots, l$.

V. Bilinear Quadratic Optimal Control

The finite horizon optimal control problem defined in (4) can now be rewritten as

$$J = \frac{1}{2}[\mathrm{y}(t_f)]^T S[\mathrm{y}(t_f)] + \frac{1}{2}\int_{t_0}^{t_f} (\mathrm{y}^T Q\mathrm{y} + \mathrm{u}^T R\mathrm{u})dt \quad (17)$$

where the symmetric positive semi-definite matrices S and Q are the finite dimensional representations of $\mathcal{S}, \mathcal{Q}$, and R is a symmetric positive-definite matrix.

By using the maximum principle, a canonical optimality condition can be obtained, which is a nonlinear two-point boundary value problem (18) and usually impossible to be solved explicitly:

$$\begin{cases} \dot{\mathrm{y}} = \dfrac{\partial \mathcal{H}}{\partial \mathrm{p}} = K\mathrm{y} - \Pi(\mathrm{y},t)R^{-1}\Pi(\mathrm{y},t)^T\mathrm{p}, \\ \dot{\mathrm{p}} = -\dfrac{\partial \mathcal{H}}{\partial \mathrm{y}} = -Q\mathrm{y} - K^T\mathrm{p} - \left(\dfrac{\partial\left[\Pi(\mathrm{y},t)\mathrm{u}\right]}{\partial \mathrm{y}}\right)^T \mathrm{p}, \\ \mathrm{u}^* = -R^{-1}\Pi(\mathrm{y},t)^T\mathrm{p}, \quad \mathrm{y}(t_0), \mathrm{p}(t_f) \text{ are given.} \end{cases} \quad (18)$$

where $\mathcal{H} = \frac{1}{2}(\mathrm{y}^T Q\mathrm{y} + \mathrm{u}^T R\mathrm{u}) + \mathrm{p}^T\left(K\mathrm{y} + \Pi(\mathrm{y},t)\mathrm{u}\right)$ and $\Pi(\mathrm{y},t)\mathrm{u} \triangleq K\mathrm{y}u_1(t) + Fu_2(t) + Gu_3(t) + Hu_4(t)$.

A convergent scheme based on quasi-linearization has been proposed in [11], [12], and references therein, to solve the optimality conditions successively. In this paper, we introduce a different scheme to solve the optimal control problem, which is based on the Picard approximation [8]. The convergence proof is similar to that of the quasi-linearization method, and makes use of contraction mapping theory [13].

A. Algorithm based on Picard Approximation

To deal with the optimal control computation of the bilinear system (16), we propose a successive approach based on the Picard approximation,

$$\begin{aligned} \dot{\mathrm{y}}^{(k+1)} = {}& K\mathrm{y}^{(k+1)} + K\mathrm{y}^{(k)}u_1^{(k+1)} \\ & + Fu_2^{(k+1)} + Gu_3^{(k+1)} + Hu_4^{(k+1)} \end{aligned} \quad (19)$$

$$\mathrm{y}^{(k)}(t_0) = \mathrm{y}_0, \quad (k = 0, 1, 2\cdots), \quad (20)$$

where the superscript (k) denotes the iteration number. We rewrite this expression as a standard linear time varying system

$$\dot{\mathrm{y}}^{(k+1)} = \mathcal{A}^{(k)}\mathrm{y}^{(k+1)} + \mathcal{B}^{(k)}(t)\mathrm{u}^{(k+1)} \quad (21)$$

$$\mathrm{y}^{(k+1)}(t_0) = \mathrm{y}^0, \quad (k = 0, 1, 2\cdots) \quad (22)$$

where $\mathcal{A}^{(k)} = \mathcal{A} = K$, $\mathcal{B}^{(k)}(t) = \left[K\mathrm{y}^{(k)}(t), F, G, H\right]$. The cost functional (17) becomes

$$\begin{aligned} J^{(k+1)} = {}& \frac{1}{2}[\mathrm{y}^{(k+1)}(t_f)]^T S[\mathrm{y}^{(k+1)}(t_f)]^T \\ & + \frac{1}{2}\int_{t_0}^{t_f}\left([\mathrm{y}^{(k+1)}]^T Q[\mathrm{y}^{(k+1)}] + [\mathrm{u}^{(k+1)}]^T R[\mathrm{u}^{(k+1)}]\right)dt. \end{aligned} \quad (23)$$

For each iteration k, we have a standard linear quadratic optimal control problem defined by (21)–(23). The closed loop system solution is given by

$$\dot{\mathrm{y}}^{(k+1)} = \left[\mathcal{A} - \mathcal{B}^{(k)}R^{-1}\mathcal{B}^{(k)^T}P^{(k+1)}\right]\mathrm{y}^{(k+1)} \quad (24)$$

with the approximate optimal control law given by

$$\mathrm{u}^{(k+1)}(t) = -R^{-1}\left(\mathcal{B}^{(k)}\right)^T P^{(k+1)}\mathrm{y}^{(k+1)}(t). \quad (25)$$

The matrix function $P^{(k+1)}(\cdot)$ is governed by the Riccati matrix differential equation

$$\begin{aligned} \dot{P}^{(k+1)} = {}& -P^{(k+1)}\mathcal{A} - \mathcal{A}^T P^{(k+1)} - Q \\ & + P^{(k+1)}\mathcal{B}^{(k)}R^{-1}(\mathcal{B}^{(k)})^T P^{(k+1)} \end{aligned} \quad (26)$$

with the terminal condition $P^{(k+1)}(t_f) = S$. Generally, an initial guess for $\mathrm{y}^{(0)}(t)$ is necessary to evaluate $\mathcal{B}^{(k)}$ in (21) and start the iterations. The procedure is stopped when the control convergence is achieved under given error tolerance. Finally, we obtain the following feedback law $\mathrm{u}^*(t)$ by using the convergent solution of the Riccati equation $P^*(t)$

$$\mathrm{u}^*(t) = -R^{-1}\mathcal{B}(t)^T P^*(t)\mathrm{y}(t), \quad (27)$$

where $\mathcal{B}(t) = (K\mathrm{y}(t), F, G, H)$.

Remark 2: In general, the LDDS based on the POD method depends on the control inputs, which requires the LDDS to be updated as the iteration evolves and the control inputs vary. However, for the problem (3)–(4), the dominant POD modes are strongly related to the system initial state and fairly independent of the control inputs. Therefore, it is not necessary to update the LDDS and control law iteratively in this particular problem.

B. Convergence Proof

In the rest of this section, it remains to prove the convergence of the proposed Picard approximation in solving the optimal control problem. Namely, we will show the following limits in appropriate functional spaces

$$\lim_{k\to\infty}\mathrm{y}^{(k)} = \mathrm{y}^*, \quad \lim_{k\to\infty}P^{(k)} = P^*. \quad (28)$$

The associated spaces are two Banach spaces (see, e.g., [11], [8]) $\mathfrak{B}_1 = C([t_0, t_f], \mathbb{R}^l), \mathfrak{B}_2 = C([t_0, t_f], \mathbb{R}^{l\times l})$, with norms

$$\|\mathrm{y}\|_{\mathfrak{B}_1} = \int_{t_0}^{t_f}\|\mathrm{y}(s)\|ds, \quad \|P\|_{\mathfrak{B}_2} = \int_{t_0}^{t_f}\|P(s)\|ds, \quad (29)$$

where $\|\mathrm{y}\| = \sqrt{\sum_{i=1}^{l}\mathrm{y}_i^2}$, $\|P\| = \sqrt{\sum_{i,j=1}^{l}P_{ij}^2}$.

Remark 3: To show (28), we only need to show that both $\left\{\mathrm{y}^{(k)}\right\}$ and $\left\{P^{(k)}\right\}$ are Cauchy sequences. Thus, the convergence follows due to the completeness of the Banach spaces. The convergence proof is based on the contraction mapping theorem for Banach spaces [13], which is motivated by the convergence proof in [11], [12].

Based on (24) and (26), we obtain differential equations for the differences $\mathrm{y}^{(k+1)} - \mathrm{y}^{(k)}$ and $P^{(k+1)} - P^{(k)}$, i.e.,

$$\begin{aligned}&\frac{d}{dt}\left[\mathrm{y}^{(k+1)} - \mathrm{y}^{(k)}\right]\\ &= \mathfrak{A}^{(k)}\left(\mathrm{y}^{(k+1)} - \mathrm{y}^{(k)}\right) + \left(\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right)\mathrm{y}^{(k)},\end{aligned} \tag{30}$$

$$\begin{aligned}&\frac{d}{dt}\left[P^{(k+1)} - P^{(k)}\right] + \left[P^{(k+1)} - P^{(k)}\right]\mathfrak{A}^{(k)}\\ &+ \mathfrak{A}^{(k-1)^T}\left[P^{(k+1)} - P^{(k)}\right] + P^{(k)}\left[\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right]\\ &+ \left[\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right]^T P^{(k+1)} + Q^{(k)} - Q^{(k-1)} = 0,\end{aligned} \tag{31}$$

where

$$\mathfrak{A}^{(k)} = \mathcal{A} - \mathcal{B}^{(k)} R^{-1}\left(\mathcal{B}^{(k)}\right)^T P^{(k+1)}, \tag{32}$$

$$Q^{(k)} = Q + P^{(k+1)}\mathcal{B}^{(k)} R^{-1}\left(\mathcal{B}^{(k)}\right)^T P^{(k+1)}. \tag{33}$$

In order to express the solutions of (30) and (31), we introduce the transition matrix $\Phi^{(k)}(t, t_0)$ which solves

$$\dot{\Phi}^{(k+1)}(t, t_0) = \mathfrak{A}^{(k)}(t)\Phi^{(k+1)}(t, t_0), \tag{34}$$

$$\Phi^{(k+1)}(t_0, t_0) = I. \tag{35}$$

In the subsequent proof we will use some of the following properties of the transition matrix $\Phi(\cdot, \cdot)$:

$$\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0), \quad \Phi^{-1}(t, s) = \Phi(s, t). \tag{36}$$

The following lemma provides solutions for (30) and (31).

Lemma 2: The solutions of (30) and (31) are

$$\mathrm{y}^{(k+1)} - \mathrm{y}^{(k)} \tag{37}$$

$$= \int_{t_0}^{t} \Phi^{(k+1)}(t, s)\left(\mathfrak{A}^{(k)}(s) - \mathfrak{A}^{(k-1)}(s)\right)\Phi^{(k)}(s, t_0)\mathrm{y}^0 ds$$

$$\begin{aligned}&P^{(k+1)} - P^{(k)} = \int_{t}^{t_f} [\Phi^{(k)}(s, t)]^T\\ &\times \left\{P^{(k)}\left[\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right] + \left[\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right]^T P^{(k+1)}\right.\\ &\left.\quad + Q^{(k)} - Q^{(k-1)}\right\}\Phi^{(k+1)}(s, t)ds.\end{aligned} \tag{38}$$

Proof: The integral expression for $\mathrm{y}^{(k+1)} - \mathrm{y}^{(k)}$ can be obtained by directly integrating both sides of the linear system (30). This expression is written in terms of the transition matrix $\Phi(t, t_0)$ defined in (34)–(36). We note that the initial value of the difference term $\mathrm{y}^{(k+1)}(t_0) - \mathrm{y}^{(k)}(t_0) = 0$ due to (22). Therefore, only the inhomogeneous term of the solution appears in (37). Additionally, we use the transition matrix to write $\mathrm{y}^{(k)}(s) = \Phi^{(k)}(s, t_0)\mathrm{y}^0$ in (37).

For the integral expression for $P^{(k+1)} - P^{(k)}$, we first use the definition (34) of the transition matrix to compute the derivative in time of $[\Phi^{(k)}(t, t_0)]^T\left[P^{(k+1)} - P^{(k)}\right]\Phi^{(k+1)}(t, t_0)$,

$$\begin{aligned}&\frac{d}{dt}\left\{[\Phi^{(k)}(t, t_0)]^T\left[P^{(k+1)} - P^{(k)}\right]\Phi^{(k+1)}(t, t_0)\right\}\\ &= [\Phi^{(k)}(t, t_0)]^T[\mathfrak{A}^{(k-1)}(t)]^T\left[P^{(k+1)} - P^{(k)}\right]\Phi^{(k+1)}(t, t_0)\\ &+ [\Phi^{(k)}(t, t_0)]^T\left[P^{(k+1)} - P^{(k)}\right]\mathfrak{A}^{(k)}(t)\Phi^{(k+1)}(t, t_0)\\ &+ [\Phi^{(k)}(t, t_0)]^T\frac{d}{dt}\left[P^{(k+1)} - P^{(k)}\right]\Phi^{(k+1)}(t, t_0).\end{aligned} \tag{39}$$

Then we use (31) to rewrite (39) as

$$\begin{aligned}&\frac{d}{dt}\left\{[\Phi^{(k)}(t, t_0)]^T\left[P^{(k+1)} - P^{(k)}\right]\Phi^{(k+1)}(t, t_0)\right\}\\ &= [\Phi^{(k)}(t, t_0)]^T\left\{\left[\mathfrak{A}^{(k-1)} - \mathfrak{A}^{(k)}\right]^T P^{(k+1)} + Q^{(k-1)}\right.\\ &\left.\quad - Q^{(k)} + P^{(k)}\left[\mathfrak{A}^{(k-1)} - \mathfrak{A}^{(k)}\right]\right\}\Phi^{(k+1)}(t, t_0).\end{aligned} \tag{40}$$

Integrating both sides from t to t_f, we can obtain

$$\begin{aligned}&[\Phi^{(k)}(t, t_0)]^T\left[P^{(k+1)} - P^{(k)}\right]\Phi^{(k+1)}(t, t_0)\\ &= \int_{t}^{t_f} [\Phi^{(k)}(s, t_0)]^T\left\{P^{(k)}\left[\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right]\right.\\ &+ \left[\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right]^T P^{(k+1)}\\ &\left.+ Q^{(k)} - Q^{(k-1)}\right\}\Phi^{(k+1)}(s, t_0)ds,\end{aligned} \tag{41}$$

where the final difference term $P^{(k+1)}(t_f) - P^{(k)}(t_f)$ vanishes due to the terminal condition $P^{(k+1)}(t_f) = S$ of (26). In order to cancel $[\Phi^{(k)}(t, t_0)]^T$ and $\Phi^{(k+1)}(t, t_0)$ in (41), we multiply both sides of the equation (41) with $[\Phi^{(k)}(t_0, t)]^T$ (from the left) and $\Phi^{(k+1)}(t_0, t)$ (from the right) respectively, and use (36) to obtain the integral expression for $P^{(k+1)} - P^{(k)}$. ■

Theorem 3: There exists an appropriate control weight matrix R, such that the sequences $\{\mathrm{y}^{(k)}(t)\}$ and $\{P^{(k)}(t)\}$ generated by (24) and (26) respectively are convergent.

Proof: Taking the $\|\cdot\|_{\mathfrak{B}}$–norm of $\mathrm{y}^{(k+1)} - \mathrm{y}^{(k)}$ and $P^{(k)} - P^{(k-1)}$ derived in Lemma 2, we have

$$\left\|\mathrm{y}^{(k+1)} - \mathrm{y}^{(k)}\right\|_{\mathfrak{B}_1} \leq \mu_1\left\|\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right\|_{\mathfrak{B}_2} \tag{42}$$

$$\begin{aligned}\left\|P^{(k+1)} - P^{(k)}\right\|_{\mathfrak{B}_2} &\leq \mu_2\left\|\mathfrak{A}^{(k)} - \mathfrak{A}^{(k-1)}\right\|_{\mathfrak{B}_2}\\ &+ \mu_3\left\|Q^{(k)} - Q^{(k-1)}\right\|_{\mathfrak{B}_2}\end{aligned} \tag{43}$$

where

$$\begin{aligned}\mu_1 &= \max_{t_0 \leq s \leq t \leq t_f}\left\|\Phi^{(k+1)}(t, s)\right\|\left\|\Phi^{(k)}(s, t_0)\right\|\|\mathrm{y}_0\|,\\ \mu_2 &= \max_{t_0 \leq t \leq s \leq t_f}\left\|\Phi^{(k)}(s, t)\right\|\left(\|P^{(k+1)}\| + \|P^{(k)}\|\right)\\ &\quad \times \left\|\Phi^{(k+1)}(s, t)\right\|,\\ \mu_3 &= \max_{t_0 \leq t \leq s \leq t_f}\left\|\Phi^{(k)}(s, t)\right\|\left\|\Phi^{(k+1)}(s, t)\right\|.\end{aligned}$$

By noting the definitions (32) and (33), and by defining $S^{(k)} \triangleq \mathcal{B}^{(k)} R^{-1} \mathcal{B}^{(k)^T}$, we obtain the following norm bounds,

$$\begin{aligned}\left\|\mathfrak{A}^{(k)}-\mathfrak{A}^{(k-1)}\right\|_{\mathfrak{B}_2} &= \left\|-S^{(k-1)}P^{(k)}+S^{(k)}P^{(k+1)}\right\|_{\mathfrak{B}_2}\\ &\leq \left\|\left(S^{(k)}-S^{(k-1)}\right)P^{(k+1)}\right\|_{\mathfrak{B}_2}\\ &\quad+\left\|S^{(k-1)}\left(P^{(k+1)}-P^{(k)}\right)\right\|_{\mathfrak{B}_2}\end{aligned} \tag{44}$$

$$\begin{aligned}&\left\|Q^{(k)}-Q^{(k-1)}\right\|_{\mathfrak{B}_2} \leq \left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2}\left\|S^{(k)}P^{(k+1)}\right\|_{\mathfrak{B}_2}\\ &\quad+\left\|P^{(k)}\right\|_{\mathfrak{B}_2}\left\|S^{(k)}-S^{(k-1)}\right\|_{\mathfrak{B}_2}\left\|P^{(k)}\right\|_{\mathfrak{B}_2}\\ &\quad+\left\|P^{(k)}S^{(k)}\right\|_{\mathfrak{B}_2}\left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2}.\end{aligned} \tag{45}$$

Now we connect the terms in (44) and (45) with the factors $\left\|\mathrm{y}^{(k+1)}-\mathrm{y}^{(k)}\right\|_{\mathfrak{B}_2}$ and $\left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2}$ to obtain

$$\begin{aligned}&\left\|S^{(k)}-S^{(k-1)}\right\|_{\mathfrak{B}_2} \leq \left\|\mathcal{B}^{(k)}-\mathcal{B}^{(k-1)}\right\|_{\mathfrak{B}_2}\left\|R^{-1}\mathcal{B}^{(k)^T}\right\|_{\mathfrak{B}_2}\\ &\quad+\left\|\mathcal{B}^{(k-1)}R^{-1}\right\|_{\mathfrak{B}_2}\left\|\mathcal{B}^{(k)^T}-\mathcal{B}^{(k-1)^T}\right\|_{\mathfrak{B}_2}\\ &\leq \frac{\left(\left\|\mathcal{B}^{(k)^T}\right\|_{\mathfrak{B}_2}+\left\|\mathcal{B}^{(k-1)}\right\|_{\mathfrak{B}_2}\right)\|K\|_{\mathfrak{B}_2}}{\|R\|}\left\|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right\|_{\mathfrak{B}_2}.\end{aligned} \tag{46}$$

Using the norm bound estimates in (42)–(46), we obtain

$$\begin{aligned}&\left\|\mathrm{y}^{(k+1)}-\mathrm{y}^{(k)}\right\|_{\mathfrak{B}_1}\\ &\leq \nu_1\left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2}+\nu_2\left\|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right\|_{\mathfrak{B}_2}\end{aligned} \tag{47}$$

where μ_1 and μ_2 are defined by $\nu_1=\mu_1\left\|S^{(k-1)}\right\|_{\mathfrak{B}_2}$, $\nu_2=\mu_1\|P^{(k+1)}\|_{\mathcal{B}_2}\frac{\left(\left\|\mathcal{B}^{(k)^T}\right\|_{\mathfrak{B}_2}+\left\|\mathcal{B}^{(k-1)}\right\|_{\mathfrak{B}_2}\right)\|K\|_{\mathfrak{B}_2}}{\|R\|}$, and

$$\begin{aligned}&\left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2}\\ &\leq \nu_3\left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2}+\nu_4\left\|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right\|_{\mathfrak{B}_2}\end{aligned} \tag{48}$$

where $\nu_3 = \mu_2\left\|S^{(k-1)}\right\|_{\mathfrak{B}_2}\mu_3\left\|S^{(k)}\right\|_{\mathfrak{B}_2}(\left\|P^{(k)}\right\|_{\mathfrak{B}_2}+\left\|P^{(k+1)}\right\|_{\mathfrak{B}_2}), \nu_4=\frac{\mu_2\nu_2}{\mu_1}+\frac{\mu_3\nu_2}{\mu_1}\frac{\left\|P^{(k)}\right\|^2_{\mathfrak{B}_2}}{\left\|P^{(k+1)}\right\|_{\mathfrak{B}_2}}$.We note that (48) can be solved with respect to $\left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2}$, i.e.,

$$\left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2} \leq \frac{\nu_4}{1-\nu_3}\left\|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right\|_{\mathfrak{B}_2}. \tag{49}$$

By substituting (49) into (47), we obtain

$$\left\|\mathrm{y}^{(k+1)}-\mathrm{y}^{(k)}\right\|_{\mathfrak{B}_1} \leq \frac{\nu_2+\nu_4(\nu_1-\nu_2)}{1-\nu_4}\left\|\mathrm{y}^{(k)}-\mathrm{y}^{(k-1)}\right\|_{\mathfrak{B}_2}.$$

If $\|R\|^{-1}$ is small enough, we can make sure that the coefficients involved are less than one, i.e., $\max\left\{\left|\frac{\nu_4}{1-\nu_3}\right|,\left|\frac{\nu_2+\nu_4(\nu_1-\nu_2)}{1-\nu_4}\right|\right\}<1$. Thus, we can conclude that both $\{P^{(k)}\}$ and $\{\mathrm{y}^{(k)}\}$ are Cauchy sequences in the associated Banach spaces, i.e., $\left\|P^{(k+1)}-P^{(k)}\right\|_{\mathfrak{B}_2} \to 0$, $\left\|\mathrm{y}^{(k+1)}-\mathrm{y}^{(k)}\right\|_{\mathfrak{B}_1} \to 0$. Due to the completeness of the Banach space, any Cauchy sequence in such a complete space is convergent, thus $\lim_{k\to\infty}P^{(k)}(t)=P^*(t)$, $\lim_{k\to\infty}\mathrm{y}^{(k)}(t)=\mathrm{y}^*(t)$. ■

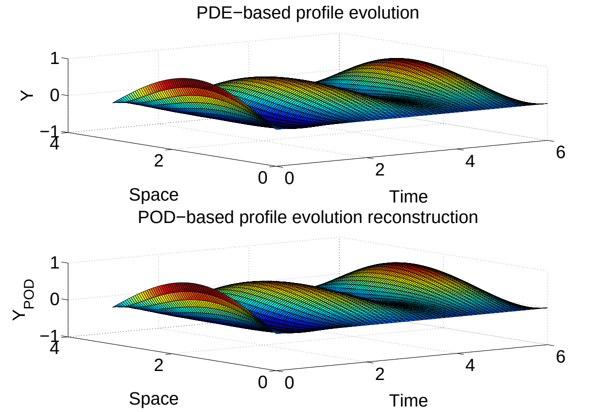

Fig. 1. Simulation of the parabolic PDE system (3) with pure Dirichlet boundary conditions using a Crank-Nicolson scheme (top); reconstruction of the profile evolution by using 7 POD modes, where the approximate error is $\|Y - Y_{POD}\|_\infty = 6.1303\times 10^{-4}$ (bottom).

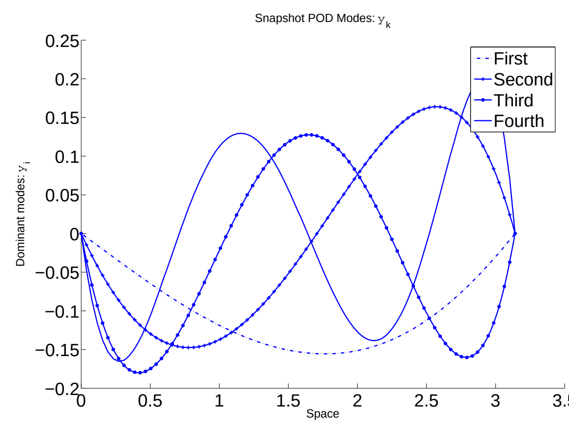

Fig. 2. The first four energetic POD modes ($l = 4$) with energy ratio $\varepsilon_4 \approx 1$ (see equation (8)); The corresponding SVD eigenvalues are $\lambda_1 = 592.3216$, $\lambda_2 = 29.8354$, $\lambda_3 = 1.9260$, $\lambda_4 = 0.0522$.

VI. Simulation Study

We enforce the diffusivity control to satisfy $|u_D(t)| < 1$ for any $t \in [t_0, t_f] = [0, 6]$. We use the Crank-Nicolson numerical scheme ($M = 60$, $N = 80$, $L = \pi$) to simulate the system (3) with the following settings: $u_D(t) = 0.01e^{-t}, \zeta(x) = 1, u_I(t) = 1-e^{-t}, \xi(x) = \frac{x}{L}\left(1-\frac{x}{L}\right), u_B^{(0)}(t) = \sin t + 0.2t, u_B^{(L)}(t) = \sin 2t, \varphi(x) = \sin x$. The system evolution and the dominant POD modes are shown in Fig. 1 and Fig. 2, respectively.

By using the first four POD modes ($l = 4$) we can construct a bilinear system with the following system matrices:

$$K=\begin{pmatrix}-1.07 & 0.40 & -0.36 & 0.29\\ 0.40 & -4.70 & 1.18 & -3.23\\ -0.36 & 1.18 & -11.30 & 1.66\\ 0.29 & -3.23 & 1.67 & -23.67\end{pmatrix}, F=\begin{pmatrix}-1.63\\ -0.16\\ -0.05\\ 0.00\end{pmatrix},$$

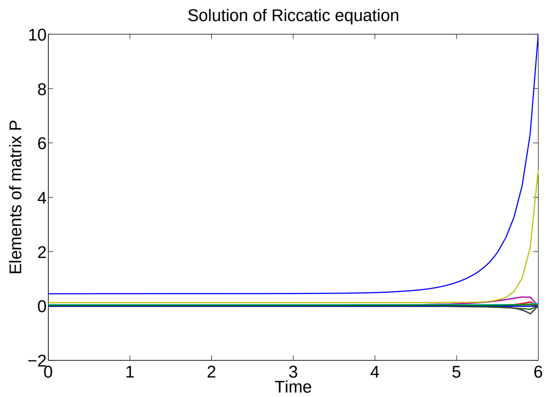

Fig. 3. Convergent solution of the Riccati equation after $k = 3$ iterations, $\|P^{(3)} - P^{(2)}\|_\infty = 1.3766 \times 10^{-4}$.

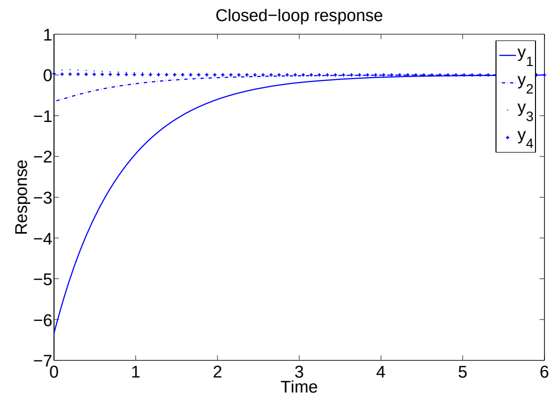

Fig. 5. System responses of the reduced order system (the third iteration).

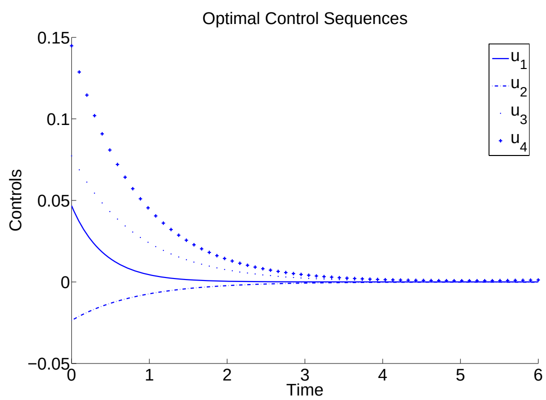

Fig. 4. Optimal controls (the third iteration).

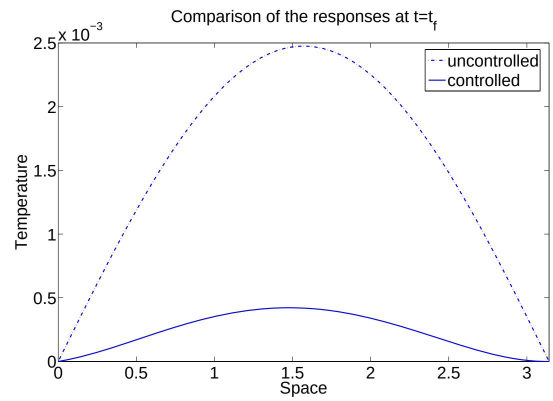

Fig. 6. Comparison of the system responses at the final time $t_f = 6$.

$$G = \begin{pmatrix} 3.85 \\ 2.33 \\ 1.79 \\ 0.97 \end{pmatrix}, H = \begin{pmatrix} 4.28 \\ -1.97 \\ 1.26 \\ -1.05 \end{pmatrix}, \mathrm{y}(0) = \begin{pmatrix} -6.33 \\ -0.65 \\ 0.06 \\ 0.03 \end{pmatrix}.$$

For the cost functional (17), we choose $Q = \mathbf{I}_{4\times4}$, $S = \mathrm{diag}(10, 5, 0.1, 0)$ and $R = \mathrm{diag}(400, 200, 150, 80)$. We use the proposed iterative scheme to compute the optimal controls. After the iteration $k = 3$, the solution of the Riccati equation converges (Fig. 3). Both the control sequences and system response are shown in Fig. 4 and Fig. 5. A comparison of the system evolutions with and without controls is shown in Fig. 6.

VII. Conclusions

In this paper we study a controlled parabolic system with three types of actuation: diffusivity, interior and boundary control. By using the POD technique, we derive a low dimensional dynamical system which governs the dominant dynamics of the original parabolic system. The reduced order system is of a bilinear form. We propose a convergent successive scheme based on the Picard approximation to compute the solution of a finite-time optimal control defined for the reduced-order bilinear system. Simulation studies show the effectiveness of the model reduction technique and the successive optimal control computation.

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