Nonlinear Robust Burn Control in Tokamaks with Uncertainties in the Fueling Lines via Lyapunov Redesign

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Abstract— **In future burning-plasma tokamaks like ITER, one of the main problems will be controlling the plasma density and temperature during long pulses in order to regulate the fusion power density. Such problem, known as burn control, requires the development of control algorithms in which modulation of the deuterium (D) and tritium (T) fueling rates may play an important role as an actuator. However, unmeasurable variations of the D-T concentration are expected in the fueling lines during such long-pulse operation. Therefore, there will be a need for robust burn controllers that can regulate the plasma density and temperature in spite of the presence of uncertainties in the D-T concentration in the fueling lines. In this work, a nonlinear controller is presented which is able to regulate the burn condition even in the presence of the aforementioned uncertainties. The controller performance is tested in simulations for a burning-plasma ITER-like scenario.**

I. Introduction

A tokamak is a torus-shaped device in which an ionized gas (plasma) is confined by means of magnetic fields in order to obtain energy from nuclear fusion reactions. In a nuclear fusion reaction, two particle nuclei fuse together, forming a bigger particle nucleus and generating energy in the process. In ITER, the next experiment in nuclear fusion research, long-pulse operation (with pulses of up to 1000 s) is planned with deuterium (D) and tritium (T), two hydrogen isotopes, as reactants. In order to achieve high-Q operation (where Q is the ratio of fusion power to auxiliary power), regulation of the plasma density and temperature around specific values will be required to sustain the associated burn condition. This control problem, known as burn control, will be of crucial importance for the success of ITER.

Traditionally, different actuators have been considered in the design of burn controllers. Modulation of the auxiliary power is a simple and efficient way to control the plasma energy [1]. Such scheme is appropriate as long as neither the minimum auxiliary power required for current drive purposes nor the maximum auxiliary power installed in the tokamak are reached. Modulation of the reactant fueling rates, normally D and T, is another actuation method considered for burn control [2]. This is a suitable method to control the plasma energy as long as disruptive density limits are not reached. In [3], a robust controller was designed based on modulation of the fueling rates to deal with uncertainties in the transport properties of the plasma particles. In [4], a nonlinear controller was proposed by combining both auxiliary power and fueling modulation with impurity injection. Also, an efficient technique to control the plasma energy by means of modulation of the fueling rates is the so-called isotopic fuel tailoring [5], in which the tritium fraction (a measure of the D-T mix) is regulated to modify the fusion power. This technique may also have other advantages, such as reducing the tritium inventory of the plasma facing components and improving the T particle confinement [6].

Although gas puffing is the main fueling technique in present-day tokamaks, it will not be the primary fueling technique in ITER because of its poor fueling efficiency due to the high plasma density and plasma-edge pressure gradient. Instead, the principal fueling technique will be pellet injection [7]. It is planned that two pellet injectors will be available in the initial phase of ITER: a D-T pellet injector with pellets of mainly T (with up to a 10%D - 90%T concentration), and a D pellet injector with pellets of 100% D. In conjunction with gas puffing (which will mainly inject D at the plasma edge), pellet injection can be employed to supply D and T at different rates. However, T is a hydrogen isotope which has a high solubility in many materials, tending to diffuse easily into the plasma facing components and tritium-processing systems. Due to such behavior, keeping a fixed D-T concentration in the fueling injectors will be a challenging problem. The T concentration may fall well below the nominal 90% in the D-T pellet injector, and a content up to 15% in T may be found in the D pellet injector. Moreover, measurements of the D-T concentrations in the fueling lines may not be available in real-time or accurate enough for control purposes. Therefore, controllers that can regulate the plasma temperature and density in the presence of unknown variations in the D-T concentrations in the fueling lines need to be developed.

In this work, a nonlinear, model-based controller for burn control is proposed. The controller synthesis makes use of modulation of the auxiliary power as the primary actuation method to regulate the plasma energy, while modulation of the fueling rates is used as backup method to control the plasma energy by means of isotopic fueling, or to directly control the plasma density when no isotopic fueling is required. In order to find a nonlinear controller that is robust to uncertainties in the D-T concentration in the fueling lines, Lyapunov redesign techniques [8] are used.

This paper is organized as follows. In Section II, the model is introduced. Section III describes the control algorithm. In Section IV, the controller is tested in simulations. Finally, some conclusions are presented in Section V.

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II. Burning Plasma Model

The model considers the existence of four different types of particles in the burning plasma: the reactants (D and T), the product of the reaction (α particles), and impurities. This is a simplified, zero-dimensional model, in which all variables represent volume-averaged magnitudes.

The balance equations for the D and T densities, n_{D} and n_{T}, are given by

$$\frac{dn_{\text{D}}}{dt} = -\frac{n_{\text{D}}}{\tau_{\text{D}}} + S_{\text{D}}^{\text{inj}} - S_\alpha, \quad \frac{dn_{\text{T}}}{dt} = -\frac{n_{\text{T}}}{\tau_{\text{T}}} + S_{\text{T}}^{\text{inj}} - S_\alpha, \quad (1)$$

where the terms $-\frac{n_{\text{D}}}{\tau_{\text{D}}}$ and $-\frac{n_{\text{T}}}{\tau_{\text{T}}}$ represent the transport of D and T particles out of the plasma core, respectively, τ_{D} and τ_{T} are the D and T confinement times, respectively, $S_{\text{D}}^{\text{inj}}$ and $S_{\text{T}}^{\text{inj}}$ are the controllable D and T injection rates, respectively, and S_α is the source of α particles arising from nuclear fusion reactions, which is given by

$$S_\alpha = n_{\text{D}} n_{\text{T}} \langle \sigma v \rangle = \gamma(1-\gamma)(n_{\text{D}} + n_{\text{T}})^2 \langle \sigma v \rangle, \qquad (2)$$

where γ is the tritium fraction, defined as $\gamma = n_{\text{T}}/(n_{\text{D}}+n_{\text{T}})$, and $\langle \sigma v \rangle$ is the cross section of the D-T reaction, which is modeled as $\langle \sigma v \rangle = \exp(\frac{a_1}{T^r} + a_2 + a_3 T + a_4 T^2 + a_5 T^3 + a_6 T^4)$, where a_i and r are constant scaling parameters [9], and T is the plasma temperature. The balance equation for the α-particle density, n_α, is given by

$$\frac{dn_\alpha}{dt} = -\frac{n_\alpha}{\tau_\alpha} + S_\alpha, \qquad (3)$$

where the term $-\frac{n_\alpha}{\tau_\alpha}$ represents the transport of α particles out of the plasma core, and τ_α is the confinement time of the α particles.

For simplicity, only one type of impurity particle is considered in this work, although a more complex model with more types of particles could be used. The time evolution of the impurity particle density, n_{I}, is given by

$$\frac{dn_{\text{I}}}{dt} = -\frac{n_{\text{I}}}{\tau_{\text{I}}} + S_{\text{I}}^{\text{sp}} + S_{\text{I}}^{\text{inj}}, \qquad (4)$$

where the term $-\frac{n_{\text{I}}}{\tau_{\text{I}}}$ represents the transport of impurities out the plasma core, τ_{I} is the confinement time of the corresponding impurity particle, S_{I}^{sp} is the source of impurities arising from sputtering, and $S_{\text{I}}^{\text{inj}}$ is the source of impurities injected for control purposes. In this work, controlled impurity injection is not considered ($S_{\text{I}}^{\text{inj}} = 0$), and S_{I}^{sp} is modeled as

$$S_{\text{I}}^{\text{sp}} = f_{\text{I}}^{\text{sp}} \left(\frac{n}{\tau_{\text{I}}} + \frac{dn}{dt} \right), \qquad (5)$$

where f_{I}^{sp} is a constant parameter, and n is the total plasma density,

$$n = n_{\text{i}} + n_{\text{e}} = 3n_\alpha + 2n_{\text{D}} + 2n_{\text{T}} + (1 + Z_{\text{I}}) n_{\text{I}}, \qquad (6)$$

where $n_{\text{i}} = n_\alpha + n_{\text{D}} + n_{\text{T}} + n_{\text{I}}$ is the ion density, and n_{e} is the electron density, which is related to the density of the ions by the quasi-neutrality condition, $n_{\text{e}} = 2n_\alpha + n_{\text{D}} + n_{\text{T}} + Z_{\text{I}} n_{\text{I}}$, where Z_{I} is the atomic number of the impurities.

The plasma energy, E, is related to n and T by

$$E = \frac{3}{2}(n_{\text{i}} T_{\text{i}} + n_{\text{e}} T_{\text{e}}) = \frac{3}{2} n T, \qquad (7)$$

where it is assumed that the ion temperature and the electron temperature are the same, $T_{\text{i}} = T_{\text{e}} = T$. The energy balance in the plasma is given by

$$\frac{dE}{dt} = -\frac{E}{\tau_E} + P \triangleq -\frac{E}{\tau_E} + P_\alpha + P_{\text{Ohm}} - P_{\text{rad}} + P_{\text{aux}}, \quad (8)$$

where τ_E is the energy confinement time, $P \triangleq P_\alpha + P_{\text{Ohm}} - P_{\text{rad}} + P_{\text{aux}}$ is the total plasma power, P_α is the α-particle heating power, P_{Ohm} is the ohmic heating power, P_{rad} is the radiative power, and P_{aux} is the auxiliary power injected to the plasma. The α-particle power is given by $P_\alpha = Q_\alpha S_\alpha$, where Q_α = 3.52 MeV. The ohmic power is given by $P_{\text{Ohm}} = 2.8 \times 10^{-9} (Z_{\text{eff}} I_{\text{p}}^2)/(a^4 T^{3/2})$, where $Z_{\text{eff}} = (4n_\alpha + n_{\text{D}} + n_{\text{T}} + Z_{\text{I}}^2 n_{\text{I}})/n_{\text{e}}$ is the effective atomic number of the plasma ions, I_{p} is the plasma current, a is the minor radius of the tokamak and T has to be given in keV. The radiative power is composed by three terms, $P_{\text{rad}} = P_{\text{brem}} + P_{\text{line}} + P_{\text{rec}}$, where P_{brem} is the Bremsstrahlung term, P_{line} is the line radiation term, and P_{rec} is the recombination term. Each term is given by $P_{\text{brem}} = 4.8 \times 10^{-37} \big(\sum_i n_i Z_i^2 \big) n_{\text{e}} \sqrt{T}$, $P_{\text{line}} = 1.8 \times 10^{-38} \big(\sum_i n_i Z_i^4 \big) n_{\text{e}} T^{-1/2}$, and $P_{\text{rec}} = 4.1 \times 10^{-40} \big(\sum_i n_i Z_i^6 \big) n_{\text{e}} T^{-3/2}$, where the summation in i is done for all types of particles in the plasma, and T has to be given in keV [10]. For τ_E, the IPB98(y,2) scaling is used [11],

$$\tau_E = H_{\text{H}} I_{\text{p}}^{0.93} B_{\text{T}}^{0.15} n_{\text{e19}}^{0.41} M^{0.19} R^{1.97} \epsilon^{0.58} \kappa_{95}^{0.78} (PV)^{-0.69},$$

where H_{H} is a constant in this model, I_{p} has to be given in MA, B_{T} is the toroidal magnetic field, n_{e19} is the electron density in 10^{19}m^{-3}, $M = 3\gamma + 2(1-\gamma)$ is the effective mass of the plasma, R is major radius of the tokamak, $\epsilon = a/R$ is the aspect ratio, κ_{95} is the elongation at the 95% flux surface/separatrix and V is the plasma volume. It is assumed that all particle confinement times scale with τ_E, i.e., $\tau_\alpha = k_\alpha \tau_E$, $\tau_{\text{D}} = k_{\text{D}} \tau_E$, $\tau_{\text{T}} = k_{\text{T}} \tau_E$, $\tau_{\text{I}} = k_{\text{I}} \tau_E$, where k_α, k_{D}, k_{T} and k_{I} are constant parameters.

In this work, it is considered that the two fueling lines available in the initial phase of ITER are employed: the D-T pellet injector with a nominal 10%D - 90%T concentration, and the D pellet injector with a nominal 100 % D concentration. Their fueling rates are denoted as $S_{\text{DT-line}}^{\text{inj}}$ and $S_{\text{D-line}}^{\text{inj}}$, respectively, and are the directly controllable. $S_{\text{D}}^{\text{inj}}$ and $S_{\text{T}}^{\text{inj}}$ can be expressed as

$$S_{\text{D}}^{\text{inj}} = (1 - \gamma_{\text{DT-line}}) S_{\text{DT-line}}^{\text{inj}} + (1 - \gamma_{\text{D-line}}) S_{\text{D-line}}^{\text{inj}}, \quad (9)$$

$$S_{\text{T}}^{\text{inj}} = \gamma_{\text{DT-line}} S_{\text{DT-line}}^{\text{inj}} + \gamma_{\text{D-line}} S_{\text{D-line}}^{\text{inj}}, \quad (10)$$

where $\gamma_{\text{DT-line}} \in [0,1]$ and $\gamma_{\text{D-line}} \in [0,1]$ are parameters that characterize the T concentration in the D-T and D pellet injectors, respectively. Therefore, in the nominal case, $\gamma_{\text{DT-line}} = \gamma_{\text{DT-line}}^{\text{nom}}$ = 0.9 and $\gamma_{\text{D-line}} = \gamma_{\text{D-line}}^{\text{nom}}$ = 0. However, as introduced above, unknown variations over time in the D-T concentrations are expected in the fueling lines. Such uncertainties are modeled as

$$\gamma_{\text{DT-line}} = \gamma_{\text{DT-line}}^{\text{nom}} + \delta_{\text{DT-line}}, \quad \gamma_{\text{D-line}} = \gamma_{\text{D-line}}^{\text{nom}} + \delta_{\text{D-line}}, \quad (11)$$

where $\delta_{\text{DT-line}}$ and $\delta_{\text{D-line}}$ are the unknown variations in the D-T concentration in the D-T and D pellet injectors, respectively. From its definition, it is found that $\delta_{\text{DT-line}} \in [-0.9, 0.1]$ and $\delta_{\text{D-line}} \in [0, 1]$, so these uncertainties are bounded. Using (9), (10) and the definitions for $\gamma_{\text{DT-line}}$ and $\gamma_{\text{D-line}}$, the balance equations (1) can be expressed compactly in matrix form as

$$\begin{bmatrix}\dot{n}_{\text{D}}\\ \dot{n}_{\text{T}}\end{bmatrix} = \begin{bmatrix}-\frac{n_{\text{D}}}{\tau_{\text{D}}} - S_\alpha\\ -\frac{n_{\text{T}}}{\tau_{\text{T}}} - S_\alpha\end{bmatrix} + \begin{bmatrix}1-\gamma^{\text{nom}}_{\text{DT-line}} & 1-\gamma^{\text{nom}}_{\text{D-line}}\\ \gamma^{\text{nom}}_{\text{DT-line}} & \gamma^{\text{nom}}_{\text{D-line}}\end{bmatrix}\begin{bmatrix}S^{\text{inj}}_{\text{DT-line}}\\ S^{\text{inj}}_{\text{D-line}}\end{bmatrix} + \begin{bmatrix}-\delta_{\text{DT-line}} & -\delta_{\text{D-line}}\\ \delta_{\text{DT-line}} & \delta_{\text{D-line}}\end{bmatrix}\begin{bmatrix}S^{\text{inj}}_{\text{DT-line}}\\ S^{\text{inj}}_{\text{D-line}}\end{bmatrix}. \quad (12)$$

III. Control Algorithm

A. Control Objective

As introduced above, the burn-control objective is to operate at working points characterized by particular values of plasma temperature and density. Those working points are defined by the equilibrium of the dynamic equations, (3), (4), (8) and (12), in the nominal case ($\delta_{\text{DT-line}} = \delta_{\text{D-line}} = 0$),

$$\begin{aligned}
0 &= -\frac{\bar{n}_\alpha}{\bar{\tau}_\alpha} + \bar{S}_\alpha, \quad 0 = -\frac{\bar{n}_{\text{I}}}{\bar{\tau}_{\text{I}}} + f^{\text{sp}}_{\text{I}}\frac{\bar{n}}{\bar{\tau}_{\text{I}}},\\
0 &= -\frac{\bar{n}_{\text{D}}}{\bar{\tau}_{\text{D}}} - \bar{S}_\alpha + (1-\gamma^{\text{nom}}_{\text{DT-line}})\bar{S}^{\text{inj}}_{\text{DT-line}} + (1-\gamma^{\text{nom}}_{\text{D-line}})\bar{S}^{\text{inj}}_{\text{D-line}},\\
0 &= -\frac{\bar{n}_{\text{T}}}{\bar{\tau}_{\text{T}}} - \bar{S}_\alpha + \gamma^{\text{nom}}_{\text{DT-line}}\bar{S}^{\text{inj}}_{\text{DT-line}} + \gamma^{\text{nom}}_{\text{D-line}}\bar{S}^{\text{inj}}_{\text{D-line}}, \quad (13)\\
0 &= -\frac{\bar{E}}{\bar{\tau}_E} + \bar{P}_\alpha + \bar{P}_{\text{Ohm}} - \bar{P}_{\text{rad}} + \bar{P}_{\text{aux}},
\end{aligned}$$

where the bar in all variables indicates equilibrium values. The set of equations (13) that characterizes the equilibrium consists of five equations and eight unknowns. Hence, it is necessary to specify three variables to solve for the equilibrium. By introducing $n_\alpha = \bar{n}_\alpha + \tilde{n}_\alpha$, $n_{\text{D}} = \bar{n}_{\text{D}} + \tilde{n}_{\text{D}}$, $n_{\text{T}} = \bar{n}_{\text{T}} + \tilde{n}_{\text{T}}$, $n_{\text{I}} = \bar{n}_{\text{I}} + \tilde{n}_{\text{I}}$ and $E = \bar{E} + \tilde{E}$, where the tilde in all variables indicates deviations with respect to the equilibrium values, (3), (4), (8) and (12) are rewritten as

$$\begin{aligned}
\dot{\tilde{n}}_\alpha &= -\frac{\bar{n}_\alpha}{\tau_\alpha} - \frac{\tilde{n}_\alpha}{\tau_\alpha} + S_\alpha, \quad \dot{\tilde{n}}_{\text{I}} = -\frac{\bar{n}_{\text{I}}}{\tau_{\text{I}}} - \frac{\tilde{n}_{\text{I}}}{\tau_{\text{I}}} + S^{\text{sp}}_{\text{I}},\\
\dot{\tilde{n}}_{\text{D}} &= -\frac{\bar{n}_{\text{D}}}{\tau_{\text{D}}} - \frac{\tilde{n}_{\text{D}}}{\tau_{\text{D}}} - S_\alpha + (1-\gamma_{\text{DT-line}})S^{\text{inj}}_{\text{DT-line}} + (1-\gamma_{\text{D-line}})S^{\text{inj}}_{\text{D-line}},\\
\dot{\tilde{n}}_{\text{T}} &= -\frac{\bar{n}_{\text{T}}}{\tau_{\text{T}}} - \frac{\tilde{n}_{\text{T}}}{\tau_{\text{T}}} - S_\alpha + \gamma_{\text{DT-line}}S^{\text{inj}}_{\text{DT-line}} + \gamma_{\text{D-line}}S^{\text{inj}}_{\text{D-line}}, \quad (14)\\
\dot{\tilde{E}} &= -\frac{\bar{E}}{\tau_E} - \frac{\tilde{E}}{\tau_E} + P_\alpha + P_{\text{Ohm}} - P_{\text{rad}} + P_{\text{aux}},
\end{aligned}$$

where $d/dt \triangleq \dot{(\,)}$. Driving (14) to zero equates to driving the system (3), (4), (8) and (12) to a particular equilibrium.

B. Nominal Control Law

In this section, a feedback controller for the nominal system ($\delta_{\text{DT-line}} = \delta_{\text{D-line}} = 0$) is designed. This nominal control law is necessary for the subsequent derivation of the robust controller via Lyapunov redesign. First, the controller attempts to regulate $\tilde{E}$ by modulating P_{aux}. By setting

$$-\frac{\bar{E}}{\tau_E} + P = -K_E\tilde{E}, \quad (15)$$

the equation for $\tilde{E}$ in system (14) is reduced to $\dot{\tilde{E}} = -\big(\frac{1}{\tau_E} + K_E\big)\tilde{E}$, where $K_E > 0$ is a design parameter, so it can be assured that $\tilde{E} \to 0$ in time as long as (15) is fulfilled [8]. Therefore, when possible, P_{aux} is set to

$$P^{\text{unsat}}_{\text{aux}} = -K_E\tilde{E} - \frac{\bar{E}}{\tau^*_E} - P_\alpha - P_{\text{Ohm}} + P_{\text{rad}}. \quad (16)$$

However, it may not be possible to set $P_{\text{aux}} = P^{\text{unsat}}_{\text{aux}}$ because there exist saturation limits, which are denoted as $P^{\text{max}}_{\text{aux}}$ and $P^{\text{min}}_{\text{aux}}$. If $P^{\text{unsat}}_{\text{aux}} > P^{\text{max}}_{\text{aux}}$, the control algorithm keeps $P_{\text{aux}} = P^{\text{max}}_{\text{aux}}$, but it cannot be assured that $\tilde{E} \to 0$. The only possible ways to cope with this limitation are either to increase $P^{\text{max}}_{\text{aux}}$ or to improve the machine parameters (I_{p}, B_{T}, etc.). On the other hand, if $P^{\text{unsat}}_{\text{aux}} < P^{\text{min}}_{\text{aux}}$, the control algorithm keeps $P_{\text{aux}} = P^{\text{min}}_{\text{aux}}$, and again it cannot be assured that $\tilde{E} \to 0$. In that case, the controller is designed to use isotopic fueling to regulate $\tilde{E}$. Control laws for $S^{\text{inj}}_{\text{D}}$ and $S^{\text{inj}}_{\text{T}}$ are sought to drive γ to a particular value, γ^*, such that $\tilde{E}$ is stabilized. Such value γ^* is obtained from (15), which is rewritten as

$$\frac{\bar{E}}{\tau_E} + P_\alpha + P^{\text{min}}_{\text{aux}} + P_{\text{Ohm}} - P_{\text{rad}} = -K_E\tilde{E}. \quad (17)$$

Using (2), it is found that $P_\alpha = Q_\alpha\gamma(1-\gamma)(n_{\text{D}}+n_{\text{T}})^2\langle\sigma v\rangle$, and solving for γ^* in (17),

$$\gamma^*(1-\gamma^*) = \frac{-K_E\tilde{E} - \bar{E}/\tau_E - P^{\text{min}}_{\text{aux}} - P_{\text{Ohm}} + P_{\text{rad}}}{Q_\alpha(n_{\text{D}}+n_{\text{T}})^2\langle\sigma v\rangle}, \quad (18)$$

where the least-tritium solution ($\gamma^* \le 0.5$) is taken. The balance equation for γ can be obtained from (1) and the definition of γ, and it is given by

$$\dot{\gamma} = \gamma(1-\gamma)\left(\frac{1}{\tau_{\text{D}}} - \frac{1}{\tau_{\text{T}}}\right) + \frac{1}{n_{\text{D}}+n_{\text{T}}}\big[-S_\alpha + S^{\text{inj}}_{\text{T}} - \gamma\big(-2S_\alpha + S^{\text{inj}}_{\text{D}} + S^{\text{inj}}_{\text{T}}\big)\big]. \quad (19)$$

By setting $S^{\text{inj}}_{\text{T}}$ as

$$S^{\text{inj, unsat}}_{\text{T}} = \frac{\gamma[-2S_\alpha + S^{\text{inj}}_{\text{D}}]}{1-\gamma} + \frac{S_\alpha + \lambda}{1-\gamma}, \quad (20)$$

where $\lambda = -(n_{\text{D}}+n_{\text{T}})\big[\gamma(1-\gamma)/\tau_{\text{D}} + \gamma^2/\tau_{\text{T}} - \xi\big]$ and ξ is an auxiliary variable, (19) becomes

$$\dot{\gamma} = -\frac{\gamma}{\tau_{\text{T}}} + \xi. \quad (21)$$

By defining $\hat{\gamma} = \gamma - \gamma^*$, and by taking $\xi = \gamma^*/\tau_{\text{T}} + \dot{\gamma}^* - K_\gamma\hat{\gamma}$, where $K_\gamma > 0$ is a design parameter, (21) is reduced to $\dot{\hat{\gamma}} = -(1/\tau_{\text{T}} + K_\gamma)\hat{\gamma}$. Therefore, $\hat{\gamma} \to 0$. For $S^{\text{inj}}_{\text{D}}$, by taking

$$S^{\text{inj, unsat}}_{\text{D}} = S_\alpha + \frac{\bar{n}_{\text{D}}}{\tau_{\text{D}}} - K_{\text{D}}\tilde{n}_{\text{D}}, \quad (22)$$

the $\tilde{n}_{\text{D}}$ equation in (14) is reduced to $\dot{\tilde{n}}_{\text{D}} = -(1/\tau_{\text{D}} + K_{\text{D}})\tilde{n}_{\text{D}}$, and it can be assured that $\tilde{n}_{\text{D}} \to 0$ in time. Therefore, $\gamma \to \gamma^*$ and $n_{\text{D}} \to \bar{n}_{\text{D}}$, and then $n_{\text{T}} \to \frac{\gamma^*}{1-\gamma^*}\bar{n}_{\text{D}} \doteq n^*_{\text{T}}$. In general, $\gamma^* \neq \bar{\gamma}$, so the value n^*_{T} is not the equilibrium value $\bar{n}_{\text{T}}$ (but is a bounded value). Therefore, it is expected that n does not go to its equilibrium value under isotopic fueling control. When possible, the controller sets $S^{\text{inj}}_{\text{D}} = S^{\text{inj, unsat}}_{\text{D}}$ and $S^{\text{inj}}_{\text{T}} =$

$S_{\rm T}^{\rm inj,\ unsat}$ as in (22) and (20). However, the fuel injectors may reach their saturated values, which are denoted as $S_{\rm D/T}^{\rm inj,max}$ and $S_{\rm D/T}^{\rm inj,min}$, and then it cannot be assured that $\tilde{E} \rightarrow 0$. In such situation, the controller attempts to regulate $\tilde{E}$ with the saturated values $S_{\rm D/T}^{\rm inj,max}$ and $S_{\rm D/T}^{\rm inj,min}$ only if $n_{\rm e} \leq n_{\rm G}$, where $n_G = I_{\rm p}/(\pi a^2) \times 10^{14} {\rm m}^3$ is the Greenwald electron density limit. If $n_e > n_G$, isotopic fueling is abandoned.

When isotopic fueling is not used, $n_{\rm T}$ is directly controlled, and (20) is substituted by

$$S_{\rm T}^{\rm inj,\ unsat} = S_\alpha + \frac{\bar{n}_{\rm T}}{\tau_{\rm T}} - K_{\rm T}\tilde{n}_{\rm T}, \tag{23}$$

where $K_{\rm T} > 0$ is a design parameter, so $\tilde{n}_{\rm T} \rightarrow 0$. The controller sets $S_{\rm D}^{\rm inj} = S_{\rm D}^{\rm inj,unsat}$ and $S_{\rm T}^{\rm inj} = S_{\rm T}^{\rm inj,unsat}$ as in (22) and (23) when possible, and in case of saturation, the minimum or maximum values are set. Barring situations in which the fueling injectors are saturated for too long periods of time, (22) and (23) assure that $\tilde{n}_{\rm D} \rightarrow 0$ and $\tilde{n}_{\rm T} \rightarrow 0$ in time. The nominal control laws for the injection rates $S_{\text{DT-line}}^{\rm inj}$ and $S_{\text{D-line}}^{\rm inj}$ are obtained from (9) and (10) by making $\delta_{\text{DT-line}} = \delta_{\text{D-line}} = 0$ and taking into account the control laws for $S_{\rm D}^{\rm inj}$ and $S_{\rm T}^{\rm inj}$, i.e., (22) and (20)/(23).

Finally, it is shown that, for the nominal system, $n_\alpha \rightarrow \bar{n}_\alpha$ and $n_{\rm I} \rightarrow \bar{n}_{\rm I}$ in time, provided that $n_{\rm D} \rightarrow \bar{n}_{\rm D}$, $n_{\rm T} \rightarrow \bar{n}_{\rm T}$ and $E \rightarrow \bar{E}$. First, by defining $\hat{n}_{\rm I} = n_{\rm I} - f_{\rm I}^{\rm sp} n$, (4) can be rewritten as $\dot{\hat{n}}_{\rm I} + f_{\rm I}^{\rm sp}\dot{n} = -\frac{\hat{n}_{\rm I} + f_{\rm I}^{\rm sp} n}{\tau_{\rm I}} + S_{\rm I}^{\rm sp}$, and using (5), it is found that $\dot{\hat{n}}_I = -\frac{\hat{n}_I}{\tau_{\rm I}}$. Then, $\hat{n}_{\rm I} \rightarrow 0$ ($\tau_{\rm I} > 0$), and $n_{\rm I} \rightarrow f_{\rm I}^{\rm sp} n$. Second, from (3), it can be noted that positive perturbations in n_α ($\tilde{n}_\alpha > 0$) decrease the first term $-n_\alpha/\tau_\alpha$. For the second term S_α, it can be noticed that as $n_{\rm D} \rightarrow \bar{n}_{\rm D}$ and $n_{\rm T} \rightarrow \bar{n}_{\rm T}$, then from (2) it is found that $S_\alpha \rightarrow \bar{n}_{\rm D}\bar{n}_{\rm T}\langle\sigma v\rangle$. For the range of interest, $\langle\sigma v\rangle$ is an increasing function of T [9]. Taking into account that $n_{\rm I} \rightarrow f_{\rm I}^{\rm sp} n$, (6) yields

$$\lim_{n_{\rm I} \rightarrow f_{\rm I}^{\rm sp} n} n = \frac{3(\bar{n}_\alpha + \tilde{n}_\alpha) + 2(\bar{n}_{\rm D} + \bar{n}_{\rm T})}{1 - f_{\rm I}^{\rm sp}(1 + Z_{\rm I})}. \tag{24}$$

It can be noted that $\tilde{n}_\alpha > 0$ implies an increase in n. Thus, using $E \rightarrow \bar{E}$, equation (7) becomes $T = \frac{\bar{E}}{\frac{3}{2}n}$, and it can be concluded that T decreases, and also that $\langle\sigma v\rangle$ decreases. Then, S_α decreases too. On the other hand, for negative perturbations in n_α ($\tilde{n}_\alpha < 0$), $-n_\alpha/\tau_\alpha$ increases, and n decreases, T increases, $\langle\sigma v\rangle$ increases and S_α increases. (3) can be rewritten as $\dot{\tilde{n}}_\alpha = -\phi_\alpha \tilde{n}_\alpha$, where ϕ_α is some positive function. As $\phi_\alpha > 0$, the α-particles density subsystem is exponentially stable. Therefore, it can be concluded that $\tilde{n}_\alpha \rightarrow 0$. To finish the proof, from (24), it can be seen that, $n \rightarrow \frac{3\bar{n}_\alpha + 2(\bar{n}_{\rm D} + \bar{n}_{\rm T})}{1 - f_{\rm I}^{\rm sp}(1 + Z_{\rm I})} = \bar{n}$, and finally, that $n_{\rm I} \rightarrow f_{\rm I}^{\rm sp}\bar{n} = \bar{n}_{\rm I}$.

C. Robust Control Law

In this section, a robust controller for the uncertain system ($\delta \neq 0$) is designed from the nominal controller using the Lyapunov redesign technique [8]. The robust control law for the modulation of $P_{\rm aux}$ is the same as the nominal control law (16), as there is no uncertainty in the energy subsystem. To design robust control laws for $S_{\rm D}^{\rm inj}$ and $S_{\rm T}^{\rm inj}$, the $\tilde{n}_{\rm D}$-$\tilde{n}_{\rm T}$ and $\tilde{n}_{\rm D}$-$\hat{\gamma}$ subsystems are rewritten in matrix form. Using $n_{\rm D} = \bar{n}_{\rm D} + \tilde{n}_{\rm D}$ and $n_{\rm T} = \bar{n}_{\rm T} + \tilde{n}_{\rm T}$, (12) becomes

$$\begin{bmatrix} \dot{\tilde{n}}_{\rm D} \\ \dot{\tilde{n}}_{\rm T} \end{bmatrix} = f + G\big[u + \delta\big], \tag{25}$$

where

$$f = \begin{bmatrix} -\frac{\bar{n}_{\rm D}}{\tau_{\rm D}} - \frac{\tilde{n}_{\rm D}}{\tau_{\rm D}} - S_\alpha \\ -\frac{\bar{n}_{\rm T}}{\tau_{\rm T}} - \frac{\tilde{n}_{\rm T}}{\tau_{\rm T}} - S_\alpha \end{bmatrix}, u = \begin{bmatrix} S_{\text{DT-line}}^{\rm inj} \\ S_{\text{D-line}}^{\rm inj} \end{bmatrix}, \delta = G^{-1}\begin{bmatrix} -\beta \\ \beta \end{bmatrix}, \tag{26}$$

$$G = \begin{bmatrix} 1 - \gamma_{\text{DT-line}}^{\rm nom} & 1 - \gamma_{\text{D-line}}^{\rm nom} \\ \gamma_{\text{DT-line}}^{\rm nom} & \gamma_{\text{D-line}}^{\rm nom} \end{bmatrix}, \beta = [\delta_{\text{DT-line}}\ \delta_{\text{D-line}}]u. \tag{27}$$

Using (9), (10), (26) and (27), (19) can be rewritten as $\dot{\hat{\gamma}} = (\gamma - \gamma^2)\frac{\tau_{\rm T} - \tau_{\rm D}}{\tau_{\rm D}\tau_{\rm T}} + \frac{(2\gamma - 1)S_\alpha}{n_{\rm DT}} + \big[\frac{-\gamma}{n_{\rm DT}}\ \ \frac{1-\gamma}{n_{\rm DT}}\big] G[u + \delta]$, where $n_{\rm DT} = n_{\rm D} + n_{\rm T}$. By combining this expression with the equation for $\tilde{n}_{\rm D}$ in (12), it is possible to write

$$\begin{bmatrix} \dot{\tilde{n}}_{\rm D} \\ \dot{\hat{\gamma}} \end{bmatrix} = f^* + G^*[u + \delta], \tag{28}$$

where

$$f^* = \begin{bmatrix} -\frac{\bar{n}_{\rm D}}{\tau_{\rm D}} - \frac{\tilde{n}_{\rm D}}{\tau_{\rm D}} - S_\alpha \\ (\gamma - \gamma^2)\frac{\tau_{\rm T} - \tau_{\rm D}}{\tau_{\rm D}\tau_{\rm T}} + \frac{(2\gamma - 1)S_\alpha}{n_{\rm DT}} \end{bmatrix}, G^* = \begin{bmatrix} 1 - \gamma_{\text{DT-line}}^{\rm nom} & 1 - \gamma_{\text{D-line}}^{\rm nom} \\ \frac{\gamma_{\text{DT-line}}^{\rm nom} - \gamma}{n_{\rm DT}} & \frac{\gamma_{\text{D-line}}^{\rm nom} - \gamma}{n_{\rm DT}} \end{bmatrix},$$

and $n_{\rm T}$ is related to $n_{\rm D}$ and γ by $n_{\rm T} = \gamma n_{\rm D}/(1 - \gamma)$. From (26) and (27), it is found that

$$\delta = \beta \begin{bmatrix} C \\ -C \end{bmatrix}, \tag{29}$$

where $C = 1/(\gamma_{\text{DT-line}}^{\rm nom} - \gamma_{\text{D-line}}^{\rm nom})$ is a constant that depends on the nominal system parameters.

1) Robust Control Law for Density Control: For the nominal $\tilde{n}_{\rm D}$-$\tilde{n}_{\rm T}$ subsystem, i.e., equation (25) with $\delta = 0$, it has been shown that $u = \psi_n \triangleq [S_{\text{DT-line}}^{\rm inj}, S_{\text{D-line}}^{\rm inj}]^T$, given by the solution of (9)-(10), where $S_{\rm D}^{\rm inj}$ and $S_{\rm T}^{\rm inj}$ are given by the control laws (22) and (23), is a stabilizing control law. The Lyapunov function $V = \frac{1}{2}\tilde{n}_{\rm D}^2 + \frac{1}{2}\tilde{n}_{\rm T}^2$ yields $\dot{V} = -\big(\frac{1}{\tau_{\rm D}} + K_{\rm D}\big)\tilde{n}_{\rm D}^2 - \big(\frac{1}{\tau_{\rm T}} + K_{\rm T}\big)\tilde{n}_{\rm T}^2$, which is < 0 for all $\tilde{n}_{\rm D}, \tilde{n}_{\rm T} \neq 0$. The robust control law for the modulation of the fueling rates u is expressed as $u = \psi_n + v$, where v is the part to be designed for robustness. First, a bound must be found for $||\delta(\psi_n + v)||$. Taking 2-norm in (29) and using the definition for β in (27), it is found that

$$\begin{aligned} ||\delta(\psi_n + v)||_2 &= \sqrt{2}|C|\big|\big|[\delta_{\text{DT-line}}\ \delta_{\text{D-line}}]\psi_n + [\delta_{\text{DT-line}}\ \delta_{\text{D-line}}]v\big|\big|_2 \\ &\leq \sqrt{2}|C|\big(\big|\big|[\delta_{\text{DT-line}}\ \delta_{\text{D-line}}]\psi_n\big|\big|_2 + \big|\big|[\delta_{\text{DT-line}}\ \delta_{\text{D-line}}]v\big|\big|_2\big) \\ &\leq \sqrt{2\big(\delta_{\text{DT-line}}^2 + \delta_{\text{D-line}}^2\big)}|C|\big(||\psi_n||_2 + ||v||_2\big), \end{aligned} \tag{30}$$

where the triangular and Cauchy-Schwarz inequalities and the 2-norm properties have been used. Furthermore, it is possible to bound $\sqrt{2(\delta_{\text{DT-line}}^2 + \delta_{\text{D-line}}^2)}$ from above by $\sqrt{2((\delta_{\text{DT-line}}^{\max})^2 + (\delta_{\text{D-line}}^{\max})^2)}$, where $\delta_{\text{DT-line}}^{\max}$ and $\delta_{\text{D-line}}^{\max}$ are the maximum uncertainties, in absolute value, in the D-T concentrations of the pellet injectors. Then, it is possible to write

$$||\delta(\psi_n + v)||_2 \leq \kappa_0\big(||\psi_n||_2 + ||v||_2\big), \tag{31}$$

where $\kappa_0 = \sqrt{2\big((\delta_{\text{DT-line}}^{\max})^2 + (\delta_{\text{D-line}}^{\max})^2\big)}|C|$. If v is taken as

$$v = -\frac{\kappa_0||\psi_n||_2}{1 - \kappa_0}\frac{w}{||w||_2}, \tag{32}$$

where $w^T = [\frac{\partial V}{\partial \tilde{n}_{\rm D}} \;\; \frac{\partial V}{\partial \tilde{n}_{\rm T}}]G$, then the closed-loop system is robustly stable [8]. The control law for u is given by

$$u = \psi_n - \frac{\kappa_0 ||\psi_n||_2}{1-\kappa_0}\frac{w}{||w||_2}. \qquad (33)$$

However, it can be noted that (33) is not defined at $\tilde{n}_{\rm D}$ = $\tilde{n}_{\rm T}$ = 0. To avoid that problem, the control law (33) is slightly modified to make it continuous. By following a similar approach to the one in [8], u is taken as

$$u = \psi_n - \frac{\kappa_0 ||\psi_n||_2}{1-\kappa_0}\frac{w}{||w||_2}, \qquad (34)$$

if $\kappa_0||\psi_n||_2||w||_2 \geq \epsilon$, and

$$u = \psi_n - \left(\frac{\kappa_0 ||\psi_n||_2}{1-\kappa_0}\right)^2\frac{w}{\epsilon}, \qquad (35)$$

if $\kappa_0||\psi_n||_2||w||_2 < \epsilon$, where $\epsilon > 0$ is a design parameter that is small. The modified laws (34) and (35) do not assure that $\tilde{n}_{\rm D} \to 0$ and $\tilde{n}_{\rm T} \to 0$ in time, but guarantee that $|\tilde{n}_{\rm D}|$ and $|\tilde{n}_{\rm T}|$ are bounded by class $\mathcal{K}$ functions of ϵ [8].

2) Robust Control Law for Isotopic Fueling: For the nominal $\tilde{n}_{\rm D}$-$\hat{\gamma}$ subsystem, i.e., equation (28) with δ = 0, it has been shown that $u = \psi_\gamma \triangleq [S^{\rm inj}_{\text{DT-line}}, S^{\rm inj}_{\text{D-line}}]^T$, given by the solution of (9)-(10) where $S^{\rm inj}_{\rm D}$ and $S^{\rm inj}_{\rm T}$ are given by the control laws (22) and (20), is a stabilizing control law. The Lyapunov function $V^* = \frac{1}{2}\tilde{n}_{\rm D}^2 + \frac{1}{2}\hat{\gamma}^2$ yields $\dot{V}^* = -\big(\frac{1}{\tau_{\rm D}} + K_{\rm D}\big)\tilde{n}_{\rm D}^2 - \big(\frac{1}{\tau_{\rm T}} + K_\gamma\big)\hat{\gamma}^2$, which is < 0 for all $\tilde{n}_{\rm D}, \hat{\gamma} \neq 0$. Then, a robust control law with a shape given by $u = \psi_\gamma + v^*$ is sought in this case. The bound for δ is given by (31) where ψ_n is substituted by ψ_γ. Therefore, by taking

$$v^* = -\frac{\kappa_0 ||\psi_\gamma||_2}{1-\kappa_0}\frac{w^*}{||w^*||_2}, \qquad (36)$$

where ${w^*}^T = [\frac{\partial V^*}{\partial \tilde{n}_{\rm D}} \frac{\partial V^*}{\partial \hat{\gamma}}]G^*$, a robust control law for isotopic fueling control is obtained which assures stability of the closed-loop system. Again, (36) is not defined at $\tilde{n}_{\rm D}$ = $\hat{\gamma}$ = 0, and a continuous, modified version is used instead. The modified control law for u is given by

$$u = \psi_\gamma - \frac{\kappa_0 ||\psi_\gamma||_2}{1-\kappa_0}\frac{w^*}{||w^*||_2}, \qquad (37)$$

if $\kappa_0||\psi_\gamma||_2||w^*||_2 \geq \epsilon^*$, and by

$$u = \psi_\gamma - \left(\frac{\kappa_0 ||\psi_\gamma||_2}{1-\kappa_0}\right)^2\frac{w^*}{\epsilon^*}, \qquad (38)$$

if $\kappa_0||\psi_\gamma||_2||w^*||_2 < \epsilon^*$, where $\epsilon^* > 0$ is again a small constant. The modified laws (37) and (38) guarantee that $|\tilde{n}_{\rm D}|$ and $|\hat{\gamma}|$ are bounded by class $\mathcal{K}$ functions of ϵ^* [8].

Finally, it can be concluded that $|\tilde{n}_\alpha|$ and $|\tilde{n}_{\rm I}|$ are bounded by class $\mathcal{K}$ functions of ϵ or ϵ^* provided that $|\tilde{n}_{\rm D}|$ and $|\tilde{n}_{\rm T}|$ are also bounded by class $\mathcal{K}$ functions of ϵ or ϵ^*, and $E \to \bar{E}$. Since it cannot be assured that $\tilde{n}_{\rm D} \to 0$ and $\tilde{n}_{\rm T} \to 0$ in time, (24) becomes

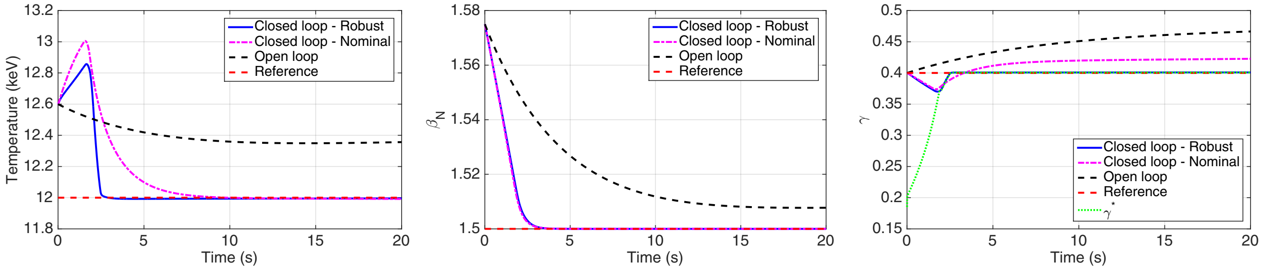
$$\lim_{n_{\rm I} \to f_{\rm I}^{\rm sp} n} n = \frac{3(\bar{n}_\alpha + \tilde{n}_\alpha) + 2(\bar{n}_{\rm D} + \bar{n}_{\rm T}) + 2(\tilde{n}_{\rm D} + \tilde{n}_{\rm T})}{1 - f_{\rm I}^{\rm sp}(1+Z_{\rm I})}. \qquad (39)$$

The proof for the density control case starts by assuming that if the robust control law keeps $|\tilde{n}_{\rm D}| < c(\epsilon)$ and $|\tilde{n}_{\rm T}| < c(\epsilon)$, then initially $|\tilde{n}_\alpha| > \frac{4}{3}c(\epsilon)$, for some class $\mathcal{K}$ function $c(\epsilon)$. Under such assumption, an increase in $\tilde{n}_\alpha$ implies an increase in n, and a decrease $\tilde{n}_\alpha$ implies a decrease in n, because regardless of how $\tilde{n}_\alpha$ influences $\tilde{n}_{\rm D}$ and $\tilde{n}_{\rm T}$, the bound imposed through $c(\epsilon)$ implies that $\tilde{n}_{\rm D}$ and $\tilde{n}_{\rm T}$ can never compensate the variations in n produced by $\tilde{n}_\alpha$. Therefore, it can be concluded that $\langle \sigma v \rangle$ decreases if $\tilde{n}_\alpha$ increases, and $\langle \sigma v \rangle$ increases if $\tilde{n}_\alpha$ decreases, when $|\tilde{n}_\alpha| > \frac{4}{3}c(\epsilon)$. For S_α, it is found that $S_\alpha = (\bar{n}_{\rm D}\bar{n}_{\rm T} + \tilde{n}_{\rm D}\bar{n}_{\rm T} + \bar{n}_{\rm D}\tilde{n}_{\rm T} + \tilde{n}_{\rm D}\tilde{n}_{\rm T})\langle \sigma v \rangle \leq (\bar{n}_{\rm D}\bar{n}_{\rm T} + |\tilde{n}_{\rm D}|\bar{n}_{\rm T} + \bar{n}_{\rm D}|\tilde{n}_{\rm T}| + |\tilde{n}_{\rm D}\tilde{n}_{\rm T}|)\langle \sigma v \rangle = (\bar{n}_{\rm D}\bar{n}_{\rm T} + b(\epsilon))\langle \sigma v \rangle$, where $b(\epsilon) = \bar{n}_{\rm D}c(\epsilon) + c(\epsilon)\bar{n}_{\rm T} + c^2(\epsilon) > 0$ is a class $\mathcal{K}$ function of ϵ because $c(\epsilon)$ is a class $\mathcal{K}$ function, and $\bar{n}_{\rm D} > 0$ and $\bar{n}_{\rm T} > 0$. As the first term in (3), $-n_\alpha/\tau_\alpha$, varies with $\tilde{n}_\alpha$ in the same way as in the proof for the nominal control law, it is possible to write $\dot{\tilde{n}}_\alpha \leq -\big(\phi_\alpha + b'(\epsilon)\big)\tilde{n}_\alpha$, where $-b'(\epsilon)\tilde{n}_\alpha = b(\epsilon)\langle \sigma v \rangle > 0$ is also a class $\mathcal{K}$ function of ϵ because $\langle \sigma v \rangle > 0$. Hence, it is possible to conclude that $\tilde{n}_\alpha \to 0$ only while $|\tilde{n}_\alpha| > \frac{4}{3}c(\epsilon)$. After some time, $|\tilde{n}_\alpha|$ decreases in such a way that $|\tilde{n}_\alpha| = \frac{4}{3}c(\epsilon)$ is reached, and then $\tilde{n}_\alpha$ may evolve in two different ways. First, it could happen that $\tilde{n}_\alpha$ decreased and remained below $|\tilde{n}_\alpha| \leq \frac{4}{3}c(\epsilon)$. On the other hand, it could happen that $|\tilde{n}_\alpha|$ grew until $|\tilde{n}_\alpha| > \frac{4}{3}c(\epsilon)$ again, but $\tilde{n}_\alpha$ cannot grow without a limit; at some point it would start decreasing again. This means that it is always possible to find a class $\mathcal{K}$ function $d(\epsilon) > \frac{4}{3}c(\epsilon)$ that bounds $|\tilde{n}_\alpha|$. To finish this proof, from (39) it can be seen that $n \to \bar{n} + \frac{3\tilde{n}_\alpha + 2(\tilde{n}_{\rm D} + \tilde{n}_{\rm T})}{1 - f_{\rm I}^{\rm sp}(1+Z_{\rm I})} \leq \bar{n} + \frac{3d(\epsilon) + 4c(\epsilon)}{1 - f_{\rm I}^{\rm sp}(1+Z_{\rm I})}$, and finally, $n_{\rm I} \to f_{\rm I}^{\rm sp} n \leq f_{\rm I}^{\rm sp}(\bar{n} + \frac{3d(\epsilon)+4c(\epsilon)}{1 - f_{\rm I}^{\rm sp}(1+Z_{\rm I})}) = \bar{n}_{\rm I} + f_{\rm I}^{\rm sp}(\frac{3d(\epsilon)+4c(\epsilon)}{1 - f_{\rm I}^{\rm sp}(1+Z_{\rm I})})$, so $\tilde{n}_{\rm I}$ is also bounded by a class $\mathcal{K}$ function of ϵ. The proof for the isotopic fueling case follows the same arguments.

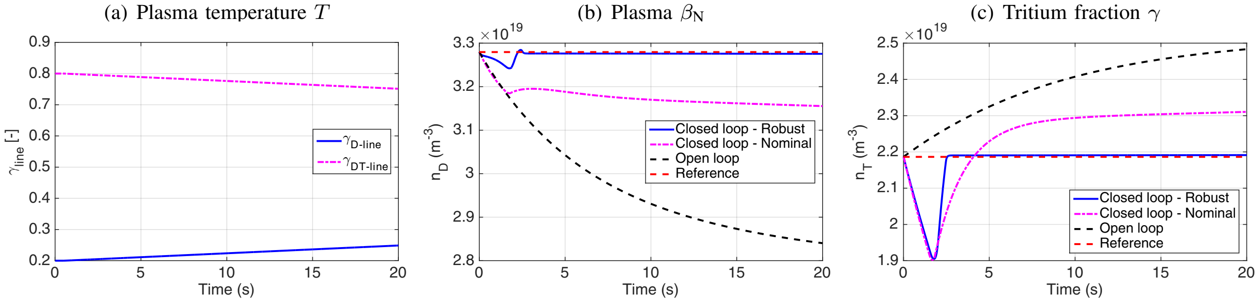
IV. Simulation Study

In this section, the controller is tested in simulations for an ITER-like scenario [11]. The machine parameters are $I_{\rm p}$ = 15 MA, $B_{\rm T}$ = 5.3 T, R = 6.2 m, a = 2 m, κ_{95} = 1.7 and V = 837 m^3. The actuation limits are $P^{\max}_{\rm aux}$ = 73/V MW m^{-3}, $P^{\min}_{\rm aux}$ = 39/V MW m^{-3}, $S^{\rm inj,min}_{\rm D/T}$ = 0, $S^{\rm inj,max}_{\rm D/T}$ = 3×10^{19} m^{-3} s^{-1}, $\dot{S}^{\max}_{\rm D/T}$ = 3×10^{19} m^{-3} s^{-2}. The confinement parameters are $H_{\rm H}$ = 1.1, k_α = 5 and $k_{\rm D}$ = $k_{\rm T}$ = 2.5. As discussed in Section II, the only impurity considered is Carbon ($Z_{\rm I}$ = 6) with $k_{\rm I}$ = 10 and $f_{\rm I}$ = 0.01. In this simulation study, the controller attempts to regulate the system around a nominal equilibrium point defined by $\bar{T}$ = 12 keV, $\bar{\gamma}$ = 0.4 and $\bar{\beta}_{\rm N}$ = 1.5, where $\beta_{\rm N} \triangleq 100\frac{4\mu_0 E a}{3 I_{\rm p} B_{\rm T}}$ ($I_{\rm p}$ must be given in MA and μ_0 is the vacuum magnetic permeability). The system starts from a perturbed initial condition of +5% in E. Also, variations in $\gamma_{\text{DT-line}}$ and $\gamma_{\text{D-line}}$ are introduced.

Fig. 1 shows the time evolution for T, $\beta_{\rm N}$, γ, $n_{\rm D}$ and $n_{\rm T}$ in open loop, in closed loop under the nominal laws, and in closed loop under the robust laws, together with the variations introduced in $\gamma_{\text{DT-line}}$ and $\gamma_{\text{D-line}}$. Fig. 2 shows the inputs to the system, $S^{\rm inj}_{\rm D}$, $S^{\rm inj}_{\rm T}$, $P_{\rm aux}$. The controller determines that isotopic fueling is required to reject the perturbation in E at the beginning of the closed-loop simulations until about $t \approx$

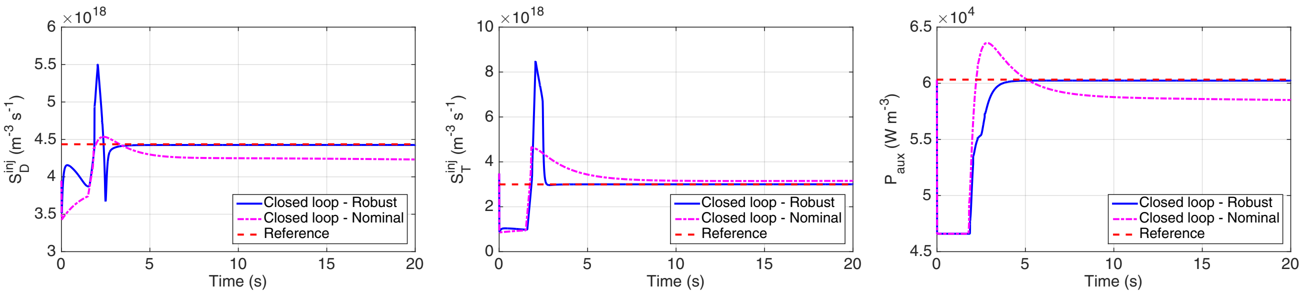


(a) Plasma temperature T (b) Plasma β_{N} (c) Tritium fraction γ



(d) D-T concentration in the fueling lines (e) Deuterium density n_{D} (f) Tritium density n_{T}

Fig. 1. Time evolution for (a) temperature T, (b) β_{N}, (c) tritium fraction γ, (d) $\gamma_{\text{D-line}}$ and $\gamma_{\text{DT-line}}$, (e) deuterium density n_{D}, and (f) tritium density n_{T}.



(a) Deuterium injection rate $S_{\mathrm{D}}^{\mathrm{inj}}$ (b) Tritium injection rate $S_{\mathrm{T}}^{\mathrm{inj}}$ (c) Auxiliary power P_{aux}

Fig. 2. Time evolution for (a) deuterium injection rate $S_{\mathrm{D}}^{\mathrm{inj}}$, (b) tritium injection rate $S_{\mathrm{T}}^{\mathrm{inj}}$ and (c) auxiliary power P_{aux}.

5 s. It can be seen that while in open loop the system evolves to a different equilibrium point, both the nominal and robust control laws drive T and β_{N} close to their equilibrium values, although the robust control law shows a faster response. However, the nominal control law cannot drive n_{D}, n_{T}, γ, $S_{\mathrm{D}}^{\mathrm{inj}}$, $S_{\mathrm{T}}^{\mathrm{inj}}$ and P_{aux} as close to their equilibrium values as driven by the robust control law. In the case of the robust control law, it can be seen that the control algorithm is "smart" enough to compensate for the unknown uncertainties and reach the nominal equilibrium.

V. Conclusions

A nonlinear-robust burn-controller has been proposed to stabilize the burn condition even in the presence of uncertainties that will most likely be found in the D-T fueling system in ITER. The model-based character of the robust controller enables a possible extension to different scenarios including recycling effects and multiple sources of impurities. This nonlinear controller is capable of rejecting large perturbations and switching between distant working points even in the presence of parametric uncertainties when other linear robust-control approaches may fail. Moreover, the controller synthesis is independent of the machine and could be extended to other tokamaks beyond ITER.

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