Simultaneous Control of Effective Atomic Number and Electron Density in Non-Burning Tokamak Plasmas

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Abstract— **The control of plasma density profiles is one of the most fundamental problems in fusion reactors. During reactor operation, the density profiles of hydrogen ions and heavier impurity ions must be precisely regulated, and while the uncontrolled open-loop behavior of these profiles is stable in non-burning plasmas, the system response time may be very slow. In this work, a controller is sought to improve the response of ion density profiles in a non-burning plasma and to track reference profiles for the electron density and effective atomic number, a set of coupled variables that are directly dependent on the hydrogen and impurity ion density profiles. A one-dimensional approximation of the transport equation for ion densities is represented in cylindrical coordinates by a partial differential equation (PDE). To control the density profiles, the PDE is discretized in space using a finite difference method and a backstepping design is applied to obtain a discrete transformation from the original system into an asymptotically stable target system. Numerical simulations of the resulting control law show that density profiles can be successfully controlled with just one step of backstepping. Tracking of electron density and effective atomic number profiles is then simulated by first transforming the profiles into corresponding ion density profiles for use by the backstepping controller. Simulations show the successful tracking of reference profiles.**

I. INTRODUCTION

To realize the promise of nuclear fusion and make it an economical energy source, tokamak reactors must be operated at inherently unstable operating points where kinetic and magnetic variables must be precisely controlled. Most approaches to the control of kinetic variables in tokamaks begin by considering 0-D (zero-dimensional) models of transport within the fusion plasma. In these models the quantities are spatially averaged over the volume of the plasma. The dynamic response of these averaged variables is then governed by ordinary differential equations (ODEs) and the problem can be approached with lumped-parameter control design techniques. In previous work the problem is simplified further by linearizing the nonlinear 0-D model and putting the model in a standard control form for which linear control techniques can be used. In [1], [2], the linearization of the model is avoided and much higher levels of performance and robustness are achieved. However, the 0-D control of the system using modulation of bulk heating, fueling and impurity density does not take into account the 1-D (one-dimensional) effect of this modulation on the spatial profiles. In a reactor, the heating, fueling, and impurity injection rates are indeed distributed throughout the plasma and effect the shape of the kinetic variable profiles.

In order to achieve high-performance control of the plasma in tokamak reactors, precise regulation of kinetic profiles is essential. The importance of these profiles stems from their effect on transport, confinement times and magneto-hydrodynamic stability within the fusion plasma. A reliable profile control system is necessary to achieve and maintain kinetic profiles that minimize transport and maximize reactor performance within stability limits. Precise profile control in experimental devices could also be useful in providing insight into the transport process and allow for conclusions to be made about the validity of proposed transport models. The importance of kinetic profile control in tokamak reactors is recognized by previous work in the field [3], [4], [5], [6], [7]. In these pieces of work, the 1-D model is represented by a set of partial differential equations (PDEs) and different methods are used to reduce the distributed parameter description of the system to a lumped parameter description. The resulting set of ODEs are linearized and conventional linear control methods are applied for the synthesis of the controller.

The objective of the controller presented in this work is to regulate both the spatial profile shape (1D) and spatial average value (0D) of the effective atomic number and the electron density in a non-burning tokamak plasma. This set of coupled variables is directly dependent on the hydrogen and impurity ion density profiles within the plasma. To control the effective atomic number and electron density, the control objective is restated in terms of the hydrogen and impurity ion density profiles through a nonlinear inversion. A controller is then sought to drive these profiles to predefined target profiles satisfying the spatial average value requirements. To control the hydrogen and impurity ion density profiles, a controller design technique capable of dealing with the distributed nature of the model is necessary.

In this work, a non-burning plasma with ion diffusion dynamics described by a 1-D PDE model is considered. The ion density profiles described by this PDE model are inherently stable. However, they are slow to converge to an equilibrium from a perturbed state. Thus, the goal of the controller is to improve the response of the system by reducing the time required for the densities to reach the desired equilibrium profiles from a set of initially perturbed profiles. To control the system, the PDE describing the density profiles is discretized in space using a finite difference method and a backstepping design is applied to obtain a discrete transformation from the original system into an asymptotically stable target system. Stability of the target

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system is achieved by the choice of convenient boundary conditions. Because the transformation is invertible for an arbitrary grid choice, it can be concluded that the discretized version of the original system is asymptotically stable and we obtain a nonlinear feedback boundary control law for the density in the original set of coordinates. This technique, which avoids linearization of the model, has already been successfully applied to other physical applications in [8], [9], [10]. Numerical simulations of the resulting control laws show that the response time of density profiles is greatly improved with just one step of backstepping, or, in other words, with a single sensor measurement of the densities from within the plasma.

The paper is organized as follows. In Section II a one dimensional PDE model that governs the dynamics of the hydrogen and impurity ion densities in a non-burning plasma is introduced. The control objective is stated in Section III. In Section IV a backstepping feedback control law that achieves asymptotic stabilization is presented. A feedback control law designed on a coarse grid is shown through a simulation study in Section V to successfully control the effective atomic number and electron density profiles of the plasma as well as their spatially averaged values. Conclusions and future work plans are stated in Section VI.

II. System Model

The electron density n_e and effective atomic number Z_{eff} are given by the following relationships

$$n_e = n_H + n_I Z_I, \tag{1}$$

$$Z_{eff} = \frac{n_H + n_I Z_I^2}{n_e}, \tag{2}$$

where n_H is the hydrogen density, n_I is the impurity ion density and Z_I is the impurity ion atomic number. The inverse transformations used to obtain the ion densities from the effective atomic number and electron density are given by

$$n_H = \frac{Z_I - Z_{eff}}{Z_I - 1} n_e, \tag{3}$$

$$n_I = \frac{Z_{eff} - 1}{Z_I(Z_I - 1)} n_e. \tag{4}$$

The 1-D model used in this work to represent the dynamics of the hydrogen and impurity ion profiles is based on an ion transport PDE in cylindrical coordinates [3]. The ion density transport equation is given by

$$\frac{\partial n}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r\left(D\frac{\partial n}{\partial r} - nV_p\right) + S, \tag{5}$$

where n is the ion density, S is the injection rate of ions and D is the particle diffusivity. In this work, the model is simplified by assuming a constant diffusivity and a negligible pinch velocity V_p. Thus, the equations for the hydrogen and impurity ion density profiles can be expressed as follows

$$\frac{\partial n_H}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD_H\frac{\partial n_H}{\partial r}\right] + S_H, \tag{6}$$

$$\frac{\partial n_I}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD_I\frac{\partial n_I}{\partial r}\right] + S_I, \tag{7}$$

where the subscripts H and I refer to hydrogen and impurity ions, respectively. The following arbitrary boundary conditions are used for the system:

$$\left.\frac{\partial n_H}{\partial r}\right|_{r=0} = 0, \tag{8}$$

$$\left.\frac{\partial n_I}{\partial r}\right|_{r=0} = 0, \tag{9}$$

$$\left.\frac{\partial n_H}{\partial r}\right|_{r=a} = k_{n_H} n_H, \tag{10}$$

$$\left.\frac{\partial n_I}{\partial r}\right|_{r=a} = k_{n_I} n_I. \tag{11}$$

III. Control Objective

At equilibrium, the hydrogen and impurity ion densities are no longer changing with respect to time and the model simplifies to a set of ODEs,

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[rD_H\frac{\partial \bar{n}_H}{\partial r}\right] + \bar{S}_H, \tag{12}$$

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left[rD_I\frac{\partial \bar{n}_I}{\partial r}\right] + \bar{S}_I. \tag{13}$$

The equilibrium profiles $\bar{n}_H$ and $\bar{n}_I$ are determined by the fueling rates $S_H(r,t) = \bar{S}_H(r)$ and $S_I(r,t) = \bar{S}_I(r)$. In this work, density actuation is only considered at the plasma's edge and the fueling rates are used only to define the equilibrium profiles. Writing $n_H(r,t) = \bar{n}_H(r) + \tilde{n}_H(r,t)$ and $n_I(r,t) = \bar{n}_I(r) + \tilde{n}_I(r,t)$, the dynamics of the deviation variables $\tilde{n}_H(r,t)$ and $\tilde{n}_I(r,t)$ are given by

$$\frac{\partial \tilde{n}_H}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD_H\frac{\partial(\bar{n}_H + \tilde{n}_H)}{\partial r}\right] + \bar{S}_H, \tag{14}$$

$$\frac{\partial \tilde{n}_I}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD_I\frac{\partial(\bar{n}_I + \tilde{n}_I)}{\partial r}\right] + \bar{S}_I. \tag{15}$$

Taking into account that

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial(\cdot)}{\partial r}\right] = \frac{\partial}{\partial r}\left[D\frac{\partial(\cdot)}{\partial r}\right] + \frac{1}{r}D\frac{\partial(\cdot)}{\partial r},$$

and noting that by (12) and (13)

$$D_H\frac{\partial^2 \bar{n}_H}{\partial r^2} + \frac{1}{r}D_H\frac{\partial \bar{n}_H}{\partial r} + \bar{S}_H = 0, \tag{16}$$

$$D_I\frac{\partial^2 \bar{n}_I}{\partial r^2} + \frac{1}{r}D_I\frac{\partial \bar{n}_I}{\partial r} + \bar{S}_I = 0, \tag{17}$$

the equations can be rewritten as

$$\frac{\partial \tilde{n}_H}{\partial t} = D_H\frac{\partial^2 \tilde{n}_H}{\partial r^2} + \frac{1}{r}D_H\frac{\partial \tilde{n}_H}{\partial r}, \tag{18}$$

$$\frac{\partial \tilde{n}_I}{\partial t} = D_I\frac{\partial^2 \tilde{n}_I}{\partial r^2} + \frac{1}{r}D_I\frac{\partial \tilde{n}_I}{\partial r}, \tag{19}$$

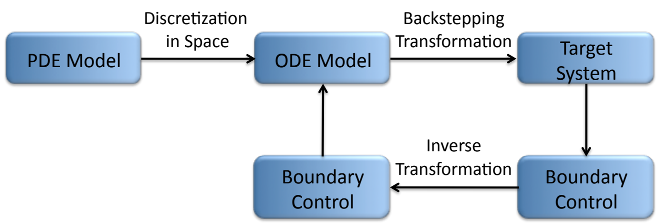

Fig. 1. Controller scheme.

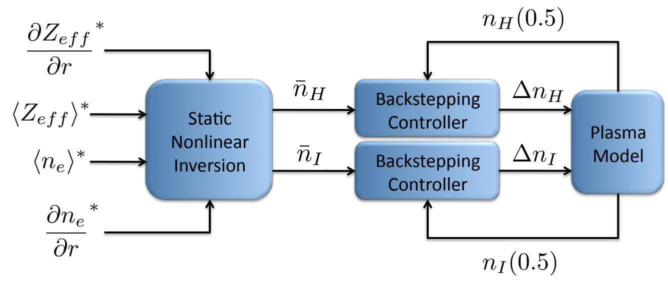

Fig. 2. Block diagram of simulation process

with boundary conditions

$$\left.\frac{\partial \tilde{n}_H}{\partial r}\right|_{r=0} = 0, \tag{20}$$

$$\left.\frac{\partial \tilde{n}_I}{\partial r}\right|_{r=0} = 0, \tag{21}$$

$$\left.\frac{\partial \tilde{n}_H}{\partial r}\right|_{r=a} = k_H \tilde{n}_H(a) + \Delta n_H(t), \tag{22}$$

$$\left.\frac{\partial \tilde{n}_I}{\partial r}\right|_{r=a} = k_I \tilde{n}_I(a) + \Delta n_I(t). \tag{23}$$

The objective of the controller is to force $\tilde{n}_H(r,t)$ and $\tilde{n}_I(r,t)$ to zero with $\Delta n_H(t)$ and $\Delta n_I(t)$ as actuation at the plasma's edge.

IV. Backstepping Technique

A backstepping technique is used to transform the original system of equations into an asymptotically stable target system. Figure 1 illustrates the controller design method. Because the hydrogen and impurity ion density profile models are completely analogous, the controller design steps are shown here for a single, generalized density profile equation. By defining $h = \frac{1}{N}$, where N is an integer, and using the notation $x_i(t) = x(ih,t)$, $i = 0, 1, ..., N$, the discretized generalized version of (18)-(19) can be written as

$$\dot{\tilde{n}}_i = D\frac{\tilde{n}_{i+1} - 2\tilde{n}_i + \tilde{n}_{i-1}}{h^2} + \frac{1}{ih}D\frac{\tilde{n}_{i+1} - \tilde{n}_i}{h}, \tag{24}$$

with the boundary conditions written as

$$\frac{\tilde{n}_1 - \tilde{n}_0}{h} = 0, \tag{25}$$

$$\frac{\tilde{n}_N - \tilde{n}_{N-1}}{h} = k_n \tilde{n} + \Delta n. \tag{26}$$

Next, an asymptotically stable target system is considered

$$\begin{aligned}\frac{\partial \tilde{F}}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{F}}{\partial r}\right] - C\tilde{F} \\ &= D\frac{\partial^2 \tilde{F}}{\partial r^2} + \frac{1}{r}D\frac{\partial \tilde{F}}{\partial r} - C\tilde{F},\end{aligned} \tag{27}$$

with $C > 0$ and the following boundary conditions

$$\left.\frac{\partial \tilde{F}}{\partial r}\right|_{r=0} = 0, \tag{28}$$

$$\left.\frac{\partial \tilde{F}}{\partial r}\right|_{r=a} = -G\tilde{F}(a). \tag{29}$$

The target system can be discretized as

$$\dot{\tilde{F}}_i = D\frac{\tilde{F}_{i+1} - 2\tilde{F}_i + \tilde{F}_{i-1}}{h^2} + \frac{1}{ih}D\frac{\tilde{F}_{i+1} - \tilde{F}_i}{h} - C\tilde{F}, \tag{30}$$

with the boundary conditions written as

$$\frac{\tilde{F}_1 - \tilde{F}_0}{h} = 0, \tag{31}$$

$$\frac{\tilde{F}_N - \tilde{F}_{N-1}}{h} = -G\tilde{F}_N. \tag{32}$$

Next, a backstepping transformation is found in the form

$$\tilde{F}_i = \tilde{n}_i - \alpha_{i-1}(\tilde{n}_1, \ldots, \tilde{n}_{i-1}). \tag{33}$$

By subtracting (30) from (24), the expression $\dot{\alpha}_{i-1} = \dot{\tilde{n}}_i - \dot{\tilde{F}}_i$ is obtained. This expression can be rearranged and rewritten in terms of $\alpha_{k-1} = \tilde{n}_k - \tilde{F}_k$, $k = i-1, i, i+1$, to obtain

$$\begin{aligned}\alpha_i =& \frac{1}{D + D/i}\left[\left(2D + \frac{D}{i} + C_F h^2\right)\alpha_{i-1} - D\alpha_{i-2}\right. \\ & \left. - h^2 C_F \tilde{n}_i + h^2 \dot{\alpha}_{i-1}\right],\end{aligned} \tag{34}$$

starting with $\alpha_0 = 0$ and where $\dot{\alpha}_{i-1}$ is calculated as

$$\dot{\alpha}_{i-1} = \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \tilde{n}_k}\dot{\tilde{n}}_k. \tag{35}$$

Next, subtracting (32) from (26) and putting the resulting equation in terms of $\alpha_{k-1} = \tilde{n}_k - \tilde{F}_k$, $k = i-1, i$ the control Δn can be defined as

$$\Delta n = \frac{\alpha_{N-1} - \alpha_{N-2}}{h} - k_n \tilde{n}_N - G(\tilde{n}_N - \alpha_{N-1}). \tag{36}$$

This equation can then be rewritten as the stabilizing law for the ion density at the plasma's edge, i.e.,

$$\tilde{n}_N = \alpha_{N-1} + \frac{1}{(1+Gh)}\left[\tilde{n}_{N-1} - \alpha_{N-2}\right]. \tag{37}$$

To show the asymptotic stability of the target system, we consider the Lyapunov function candidate

$$V = \frac{1}{2}\int_0^a r\tilde{F}^2 dr.$$

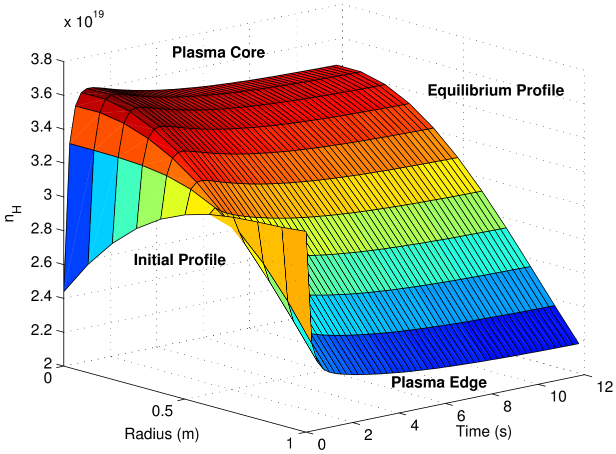

Fig. 3. Open-loop time evolution of hydrogen ion density.

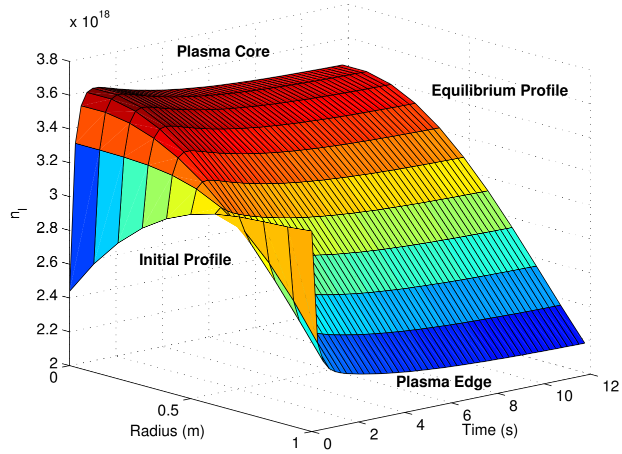

Fig. 4. Open-loop time evolution of impurity ion density.

Taking the derivative of this function with respect to time gives

$$\begin{aligned}\dot{V} &= \int_0^a r\tilde{F}\dot{\tilde{F}}dr \\ &= \int_0^a r\tilde{F}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \tilde{F}}{\partial r}\right] - C\tilde{F}\right\}dr \\ &= \tilde{F}rD\frac{\partial \tilde{F}}{\partial r}\bigg|_0^a - \int_0^a rD\left(\frac{\partial \tilde{F}}{\partial r}\right)^2 dr - \int_0^a rC\tilde{F}^2 dr \\ &= aD\tilde{F}(a)\tilde{F}_r(a) - \int_0^a rC\tilde{F}^2 dr - \int_0^a rD\tilde{F}_r^2 dr,\end{aligned}$$

where the operation $\frac{\partial(\cdot)}{\partial r}$ is denoted by $(\cdot)_r$. Using the boundary condition (29), this can be written as

$$\dot{V} = -GaD\tilde{F}(a)^2 - \int_0^a rC\tilde{F}^2 dr - \int_0^a rD\tilde{F}_r^2 dr. \quad (38)$$

It can be concluded that

$$\dot{V} \leq -CV - GaD\tilde{F}(a)^2 - \int_0^a rD\tilde{F}_r^2 dr, \quad (39)$$

and, because

$$GaD\tilde{F}(a)^2 \geq 0, \quad (40)$$

$$\int_0^a rD\tilde{F}_r^2 dr \geq 0 \quad (41)$$

we can conclude that $\dot{V} \leq -CV$, proving the asymptotic stability of the system.

The control strategy is summarized in Figure 2. Given the desired shapes ($\partial/\partial r$) and spatially averaged values ($<>$) for the effective atomic number and electron density (indicated by the star notation in the figure), equilibrium profiles are computed satisfying these specifications. These equilibrium profiles are then converted to corresponding hydrogen and impurity ion density equilibrium profiles ($\bar{n}_H$, $\bar{n}_I$) using the inverse transformations (3) and (4). The set of equilibrium profiles are then utilized by the backstepping controller to actuate the ion densities at the plasma edge in order to achieve the desired shapes and spatially averaged values for the electron density and effective atomic. We will show in the next section that single measures of the ion densities at $r = 0.5$ are enough to regulate the plasma profiles.

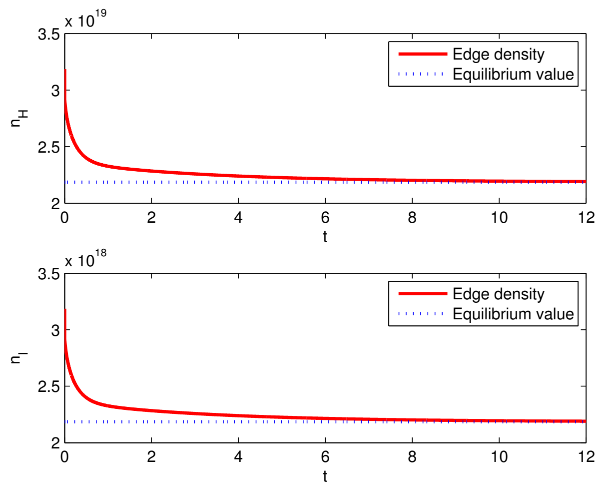

Fig. 5. Open-loop edge density response.

V. Simulations

The simulation results in this section were obtained using a spatial step size of $h = h_s = 0.1$, while the controller only uses one step of backstepping, or a spatial step size of $h = h_c = 0.5$ for a plasma of $a = 1$. The choice of the simulation grid size was based on the standard guidelines for stability of the numerical method used, while the choice of the backstepping grid size was motivated by the goal of limiting the number of sensors needed, thereby minimizing the cost of implementing the controller. The results shown reflect a non-burning plasma with the quadratic fueling profiles $\bar{S}_H = S_{H_0}[1-(r/a)^2]$ and $\bar{S}_I = S_{I_0}[1-(r/a)^2]$ and initially

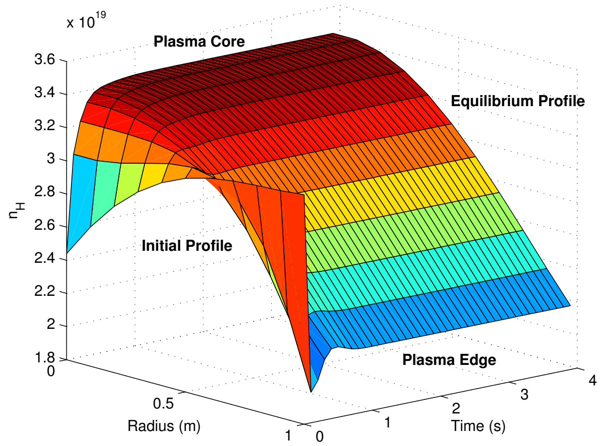

Fig. 6. Time evolution of hydrogen ion density.

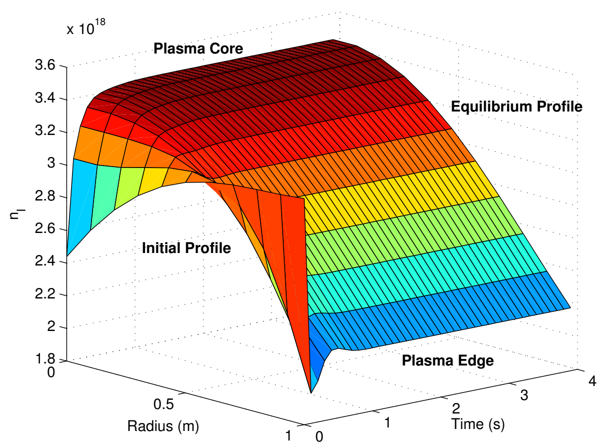

Fig. 7. Time evolution of impurity ion density.

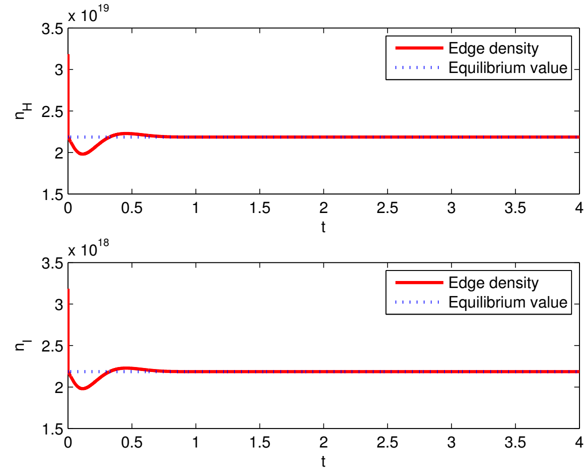

Fig. 8. Edge modulation for hydrogen and impurity ion density profile.

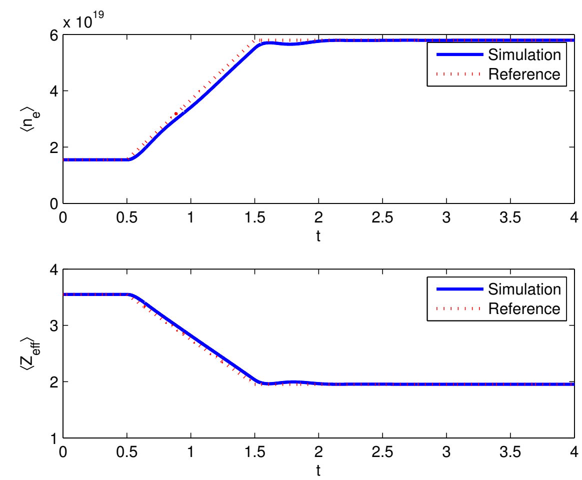

Fig. 9. Effective atomic number and electron density response (spatial averages).

perturbed density profiles $\tilde{n}_H(r,0) = (-1+2r/a)10^{19}$ and $\tilde{n}_I(r,0) = (-1+2r/a)10^{18}$. The simulation parameters are $D = 0$ and $Z_I = 6$. For the first set of simulations (Fig. 3–Fig. 8) we have used $S_{H_0} = 1.5\times10^{19}$ and $S_{I_0} = 1.5\times10^{18}$, while for the second set of simulations (Fig. 9–Fig. 13) we have used $S_{H_0} = 2.8\times10^{18}$ and $S_{I_0} = 4.5\times10^{17}$. For these conditions, the open-loop response, shown in Figures 3 and 4 is stable, however the rate of convergence from the initially perturbed profiles to the equilibrium profiles is quite slow. Figure 5 shows the open-loop response of the ion densities at the edge of the plasma and indicates that the system takes several seconds to settle to equilibrium. The closed-loop simulation results in Figures 6 and 7 show that with just one measurement from the plasma interior, the boundary controller is able to improve the system response over that of the open-loop system. The plot of closed-loop edge densities in Figure 8 shows that the response time improvement is roughly of an order of magnitude.

Figure 9 shows the response of the spatially averaged effective atomic number and electron density against the established time-varying references. The controller successfully tracks the references and the system reaches equilibrium quickly. Figure 10 plots the spatial averages of hydrogen and impurity ion densities against the references obtained from the inverse transformations (3) and (4). The edge modulation generated by the controller is found in Figure 11. The effect of this modulation on the 1-D ion profiles is shown in the plots in Figures 12 and 13.

VI. Conclusions and Future Work

A feedback controller based on Lyapunov backstepping design that improves the response of the density profiles in a cylindrical plasma has been designed. The resulting controller holds for any finite discretization in space of the original PDE model and the simulation in this work shows that a controller using just one step of backstepping successfully controls the density profiles. This controller is then shown to be capable of tracking time-varying reference for the spatially averaged effective atomic number and electron density, a set of coupled variables that are directly dependent on the hydrogen and impurity ion densities.

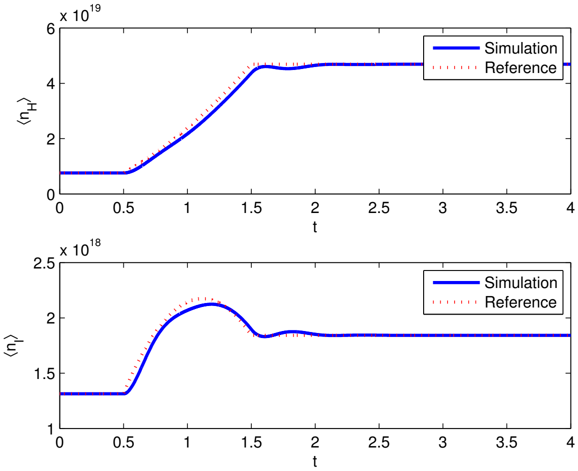

Fig. 10. Ion density response (spatial averages).

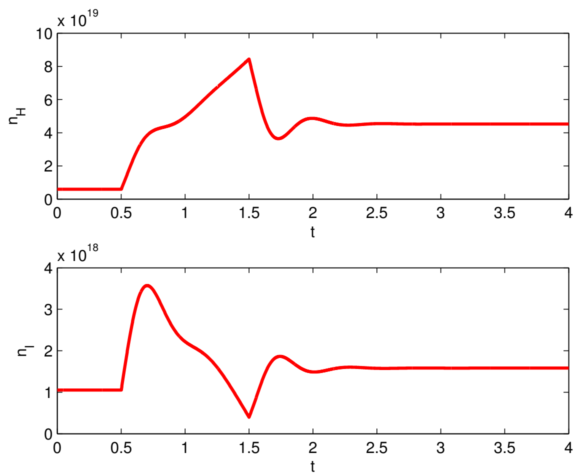

Fig. 11. Edge modulation of hydrogen and impurity ion densities.

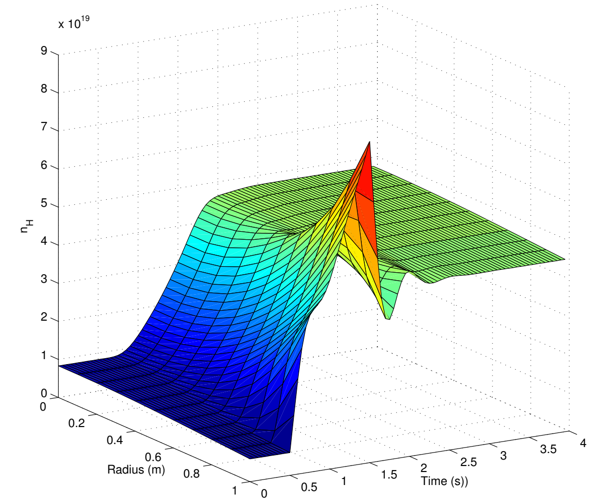

Fig. 12. Time evolution of hydrogen ion density.

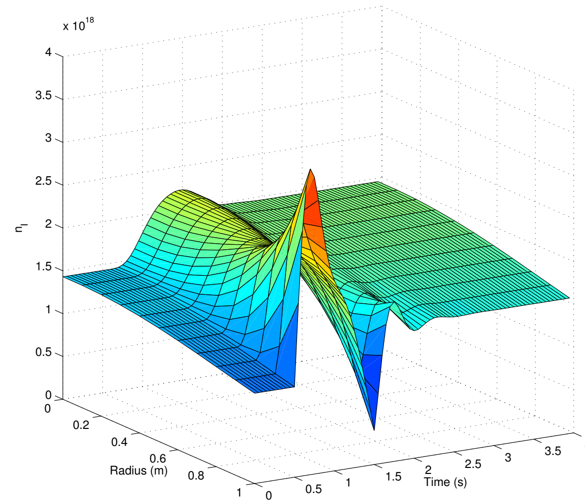

Fig. 13. Time evolution of impurity ion density.

In the future, exploiting the capability of the proposed backstepping technique to deal with nonlinear PDE systems, improvements to the system model will be made, including use of a non-constant value for the diffusion coefficient D. In addition, we will consider a burning plasma in order to study the effectiveness of using this technique to control one-dimensional density and temperature profiles in an inherently thermally unstable plasma.

REFERENCES

[1] E. Schuster, M. Krstic and G. Tynan, "Burn Control in Fusion Reactors via Nonlinear Stabilization Techniques", *Fusion Science and Technology*, vol. 43, no. 1, pp. 18-37, 2003.

[2] E. Schuster, M. Krstic and G. Tynan, Nonlinear Lyapunov-Based Burn Control in Fusion Reactors, *Fusion Engineering and Design*, 63-64, pp.569-575, 2002.

[3] M. A. Firestone and C. E. Kessel, "Plasma Kinetics Control in a Tokamak", *IEEE Transactions on Plasma Science*, vol.19, no.1, pp. 29-41, 1991.

[4] G. H. Miley and V. Varadarajan, "On Self-tuning Control of Tokamak Thermokinetics", *Fusion Technology*, vol.22, pp. 425-38, 1992.

[5] V. Fuchs, M. M. Shoucri, G. Thibaudeau, L. Harten and A. Bers, "High-Q Thermally Stable Operation of a Tokamak Reactor", *IEEE Transactions on Plasma Science*, vol. PS-11, no.1, pp. 4-18, March 1983.

[6] M. A. Firestone, J. W. Morrow-Jones and T. K. Mau, "Comprehensive Feedback Control of a Tokamak Fusion Reactor", *Fusion Technology*, vol.32, pp. 390-403, Nov. 1997.

[7] M. A. Firestone, J. W. Morrow-Jones and T. K. Mau, "Developing Integrated Tokamak Dynamics Models for Next Generation Machine Control", *Fusion Technology*, vol.32, pp. 526-544, Dec. 1997.

[8] E. Schuster, M. Krstic, "Control of a Non-linear PDE System arising from Non-burning Tokamak Plasma Transport Dynamics", *International Journal of Control*, vol. 76, no. 11, pp. 1116-2224, 2003.

[9] D. Boskovic and M. Krstic, "Backstepping Control of a Chemical Tubular Reactor", *Computers and Chemical Engineering*, vol. 26, no. 7-8, pp. 1077-1085, 2002.

[10] D. Boskovic and M. Krstic, "Stabilization of a Solid Propellant Rocket Instability by State Feedback", *Int. J. Robust Nonlinear Control*, vol. 13, pp. 483495, 2003.