LA-UR-04-5525

# AUTOMATED BEAM STEERING USING OPTIMAL CONTROL*

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*Abstract*

We present a steering algorithm which, with the aid of a model, allows the user to specify beam behavior throughout a beamline, rather than just at specified beam position monitor (BPM) locations. The model is used primarily to compute the values of the beam phase vectors from BPM measurements, and to define cost functions that describe the steering objectives. The steering problem is formulated as constrained optimization problem; however, by applying optimal control theory we can reduce it to an unconstrained optimization whose dimension is the number of control signals.

## INTRODUCTION

Steering and orbit correction for charged particle beams has been extensively studied and many laboratories have developed successful algorithms for these applications. Some of the most popular (and effective) techniques are the MICADO [2] and SVD [5], and virtual BPM [4]. Some techniques require access only to actuator settings (e.g., steering magnets) and sensor signals (e.g., BPMs), while others employ some type of model or response matrix. The objective of almost all steering algorithms is to minimize the RMS error between the measured beam positions and the design trajectory, that is, to put the beam on the design axis or hit a target position.

Here we present an alternate steering algorithm which is essentially a dual to the response matrix approach. Whereas response matrices forward propagate actuator strengths to the changes in the beam position, the current technique backward propagates changes in the beam position back to the actuators. The advantage here is that, given a beamline model, we may specify performance conditions upon the beam trajectory between BPM locations. That is, the steering algorithm considers beam behavior throughout the entire beamline. For example, we may require that the beam maintains proximity to the design axis, or specify that the beam is not steered too abruptly, or use some combination of either objective.

We formulate the beam steering objective as an optimal controls problem. First we develop the beamline model based upon transfer functions between measurement positions. We then describe a performance objective as a cost functional of the beam states and actuator strengths. This functional contains user-specified tuning parameters that describe what he or she considers an optimally steered beam. Optimal control theory then provides a theoretical framework to solve the problem.

The major drawback in this approach is that we require the full beam state vector at each measurement location. Specifically, we require the momentum coordinates as well as the position coordinates. Measurements from BPMs provide only the position coordinates. However, there are methods, known as *state observers* [6], for constructing the momentum coordinates from multiple BPM measurements (at least approximately). Due to space constraints we do not cover these techniques and simply assume we have access to the full beam states.

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## DYNAMICS MODEL

In the beamline model the *state* of the beam $\mathbf{z}(s)$, at axial location $s$ is represented by an element of phase space. Somewhat unconventionally, we parameterize phase space using *homogeneous coordinates* in $\Re^6\times\{1\}$. An element $\mathbf{z}$ in phase space is represented as

$$\mathbf{z} \equiv \begin{pmatrix} x & x' & y & y' & z & z' & 1 \end{pmatrix}^T , \tag{1}$$

where the prime indicates differentiation with respect to the path length parameter $s$, and $x,y,z$ are the position coordinates of the beam. Note that homogeneous coordinates contain an additional constant component with value 1. This approach has the advantage that translation, rotation, and scaling in phase space can all be performed by matrix multiplication. In particular, the effect of a steering magnet can be represented as a matrix action on the beam state $\mathbf{z}$ (see [1]).

We now divide the beamline into *stages* corresponding to the contiguous sections of beamline between measurement locations. Letting $s_n$ be the axial position of the entrance to stage $n$, define the states $\mathbf{z}_n = \mathbf{z}(s_n)$ for $n = 0,1\ldots N$, where $N$ is the number of stages. Each stage $n$ is then represented by a transfer function $\mathbf{F}_n(\mathbf{z}_n,\mathbf{u}_n)$ of the beam state $\mathbf{z}_n$ and the control vector $\mathbf{u}_n$. The control vector represents any actuators within the stage (e.g., steering magnets). The beam state vectors are propagated according to the transfer equations

$$\mathbf{z}_{n+1} = \mathbf{F}_n(\mathbf{z}_n, \mathbf{u}_n) \qquad n = 0,1,\ldots,N-1 \tag{2}$$

where the first state $\mathbf{z}_0$ is given. In controls parlance, this is the model of a *multistage control network*. Although the transfer functions $\{\mathbf{F}_n\}$ may include higher-order dynamics, most beamline stages can be modeled accurately enough as a transfer matrix $\mathbf{\Phi}_n(\mathbf{u}_n) \in \Re^{7\times 7}$.

## THE STEERING PROBLEM

The idea here is to provide a performance object associated with each stage $n$. This objective is embodied with a positive *cost functional* $J_n(\mathbf{z}_n,\mathbf{u}_n)$. Minimizing $J_n(\mathbf{z}_n,\mathbf{u}_n)$ gets us closer to our objective. The crux is to prescribe a $J_n$ general enough to accommodate our steering objectives, yet not so complicated as to be impractical. The functional we propose has the form

$$J_n(\mathbf{z}_n,\mathbf{u}_n) \equiv \frac{1}{2}\int_{s_n}^{s_{n+1}} \mathbf{z}(s)^T\ \mathbf{Q}\mathbf{z}(s)ds\ , \qquad (3)$$

where $\mathbf{Q} \in \Re^{7\times7}$ is a positive matrix of tuning parameters and $\mathbf{z}(s)$ is the (continuous) state vector within the stage. Clearly the state $\mathbf{z}(s)$ must be determined from a beamline model given the state $\mathbf{z}_n$ at the stage $n$ and the control signals $\mathbf{u}_n$ to the stage.

To demonstrate the computation of a meaningful merit functional $J_n$ we present that for a drift space. Its transfer function is a matrix-vector product involving only the state vector $\mathbf{z}_n$; it is the simplest meaningful example we can present without getting mired in large algebraic expressions. To further reduce the analysis we compute $J_n$ only for the $x$ plane. Thus, neglecting the homogeneous coordinate, our reduced state variable is $\mathbf{x}_n = (x_n\ x_n')^T$ and $\mathbf{x}(s)$ is given by

$$\mathbf{x}(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_n \\ x_n' \end{pmatrix}. \qquad (4)$$

Substituting the above into Eq. (3) yields the expression for the partial cost functional $J_d$ for the drift

$$\begin{aligned} J_d &= \frac{1}{2}\int_0^{l_d} \begin{pmatrix} x_n \\ x_n' \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_n \\ x_n' \end{pmatrix} ds, \\ &= \frac{q_{11}}{2} l_d x_n^2 + [\frac{q_{11}}{2} l_d^2 + \frac{1}{2}(q_{12}+q_{21})l_d] x_n x_n' \qquad (5) \\ &\quad + \frac{q_{11}}{6} l_d^3 + \frac{1}{4}(q_{12}+q_{21})l_d^2 + \frac{q_{22}}{2} l_d ] x_n'^2\ , \end{aligned}$$

where $l_d$ is the length of the drift.

It is common in accelerator applications to also require conditions on the final state $\mathbf{z}_N$. For example, we may want to insure that the beam hits a specified target location or interaction point, represented by the target state $\mathbf{z}_f$. This terminal objective may also be described by a cost functional, say $\phi(\mathbf{z}_N)$. Given our target state $\mathbf{z}_f$, an appropriate form for $\phi(\mathbf{z}_N)$ is given by

$$\phi(\mathbf{z}_N) \equiv \left(\mathbf{z}_N - \mathbf{z}_f\right)^T \mathbf{P}\left(\mathbf{z}_N - \mathbf{z}_f\right), \qquad (6)$$

where $\mathbf{P} \in \Re^{7\times7}$ is another positive matrix.

The complete steering objective is found by summing all the partial cost functionals $\{J_n\}$ and $\phi$ to form the total cost functional $J$ for the beamline. We have

$$J \equiv \sum_{n=0}^{N-1} J_n(\mathbf{z}_n,\mathbf{u}_n) + \phi(\mathbf{z}_N)\ . \qquad (7)$$

Note that $J$ then depends upon the full set of beamline states $\{\mathbf{z}_0,\mathbf{z}_1,\ldots,\mathbf{z}_N\}$ and controls $\{\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_{N-1}\}$.

We now formally state our steering problem: given an initial state vector $\mathbf{z}_i$ at the entrance of our system, find the set of controls $\{\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_{N-1}\}$ minimizing the objective functional $J$ while satisfying the dynamics of Eqs (2), or

$$\begin{aligned} &\min_{\{\mathbf{u}_0,\ldots,\mathbf{u}_{N-1}\}} \sum_{n=0}^{N-1} J_n(\mathbf{z}_n,\mathbf{u}_n) + \phi(\mathbf{z}_N) \\ &\text{such that } \mathbf{z}_{n+1} = \mathbf{F}_n(\mathbf{z}_n,\mathbf{u}_n) \quad \text{for } n = 0,\ldots,N-1 \qquad (8) \\ &\qquad \mathbf{z}_0 = \mathbf{z}_i \end{aligned}$$

which is the formal mathematical statement. By selecting the matrices **P** and **Q**, and their relative magnitudes, we can stipulate different performance objectives for our beam steering algorithm. We demonstrate this concept in the simulation results section.

## OPTIMAL CONTROL

Optimal control problems are formally analogous to the classical mechanics of physics. They share the same mathematical structure and can be analyzed using Hamiltonian formalism, symplectic geometry, and other tools born out of mechanics. In this analogy the potential energy of the system is given by the merit functional $J$ while the kinetic energy involves the state equations. To proceed we require the introduction of a set of *costate vectors* $\{\mathbf{p}_0,\ldots,\mathbf{p}_N\}$ analogous to the conjugate momentum of Hamilton mechanics. We now define a Hamiltonian $H_n(\mathbf{z}_n,\mathbf{p}_{n+1},\mathbf{u}_n)$, which is a function of "position" $n$, as

$$H_n(\mathbf{z}_n,\mathbf{p}_{n+1},\mathbf{u}_n) \equiv J_n(\mathbf{z}_n,\mathbf{u}_n) + \mathbf{p}_{n+1}{}^T\mathbf{F}_n(\mathbf{z}_n,\mathbf{u}_n)\ . \qquad (9)$$

From the optimal control theory, necessary conditions for solutions to problem (8) are given by the following [3]:

$$\begin{aligned} &\mathbf{z}_{n+1} = \mathbf{F}_n(\mathbf{z}_n,\mathbf{u}_n) \qquad n = 0,\ldots,N-1, \\ &\mathbf{z}_0 = \mathbf{z}_i\ , \\ &\mathbf{p}_n = \frac{\partial \mathbf{F}_n}{\partial \mathbf{z}_n}^T \mathbf{p}_{n+1} + \frac{\partial J_n}{\partial \mathbf{z}_n}, \quad n = N-1,\ldots,0, \qquad (10) \\ &\mathbf{p}_N = \frac{\partial \phi(\mathbf{z}_N)}{\partial \mathbf{z}_N}, \\ &\mathbf{u}_n = \arg\min_{\mathbf{v}} H_n(\mathbf{z}_n,\mathbf{p}_{n+1},\mathbf{v}) \quad n = 0,\ldots,N-1, \end{aligned}$$

The above equations for $\{\mathbf{z}_n\}$ and $\{\mathbf{p}_n\}$ are the discrete version of Hamilton's equations, while the final relation is known as Pontryagin's minimum principle. Together they characterize the optimal control set $\{\mathbf{u}_n\}$. Note that the state vectors $\{\mathbf{z}_n\}$ propagate forward according to action of the transfer functions $\{\mathbf{F}_n\}$ while the costate vectors $\{\mathbf{p}_n\}$ propagate backwards (covariantly). Specifically, the costates are one-forms that are "pulled back" by the differential mappings $\{\mathbf{F}_n\}$, with the addition of an inhomogeneous term $\partial J_n/\partial \mathbf{z}_n$ (also a one-form) reflecting the degree by which we "missed" our objective $J_n$.

### *The Steering Algorithm*

Although insightful, the above system can be difficult to solve. Fortunately it is not necessary. It can be shown that, for any stage $n$, the gradient of the total objective functional $J$ with respect to the control $\mathbf{u}_n$ is given by

$$\frac{\partial J}{\partial \mathbf{u}_n} = \frac{\partial H_n(\mathbf{z}_n,\mathbf{p}_{n+1},\mathbf{u}_n)}{\partial \mathbf{u}_n} = \mathbf{p}_{n+1}{}^T\frac{\partial \mathbf{F}_n}{\partial \mathbf{u}_n} + \frac{\partial J_n}{\partial \mathbf{u}_n}, \qquad (11)$$

if $\{\mathbf{z}_n\}$ and $\{\mathbf{p}_n\}$ satisfy Hamilton's equations in (10). With the gradients $\{\partial J/\partial \mathbf{u}_n\}$ we may employ any standard unconstrained optimization technique to minimize $J$ and, consequently, solve our steering problem. Thus, we have all the ingredients for a practical steering algorithm. Specifically, for some given error tolerance $\varepsilon$ our steering algorithm is outlined in Algorithm 1. We have found that this technique is much faster and more accurate than attempting to numerically compute the gradients $\{\partial J/\partial \mathbf{u}_n\}$.

```
while J > ε :    forward propagate the {z_n};
                 backward propagate the {p_n};
                 compute the {∂J/∂u_n};
                 update the control vectors {u_n};
                 compute J;
```

Algorithm 1: optimal steering algorithm

## SIMULATION RESULTS

To verify our steering algorithm we applied it to a model beamline consisting of an initial drift followed by a FODO lattice of 20 periods. Steering magnets were placed at the center of each quadrupole. The drift lengths were 14.88 cm while the quadrupole lengths were 6.10 cm. Quadrupole strengths were set for phase advance of 90 degrees. We assume access to the beam states $\{\mathbf{z}_n\}$ at each drift. For simplicity we considered only the $x$ phase plane. In each case the beam enters the channel with an offset of 0.3 cm, that is, $\mathbf{x}_i = (0.003\ 0)^T$ and our target final state is $\mathbf{x}_f = (0\ 0)^T$. We implemented the Polak-Ribiere variant of the nonlinear conjugate-gradient algorithm in conjunction with Armijo's rule [7].

Four cases were run: each identified by the tuning matrices $\{\mathbf{P}_1,\mathbf{Q}_1\}$, $\{\mathbf{P}_2,\mathbf{Q}_2\}$, $\{\mathbf{P}_3,\mathbf{Q}_3\}$, and $\{\mathbf{P}_4,\mathbf{Q}_4\}$. The first two cases are shown in Figure 1. In these two cases only terminal condition were enforced, that is $\mathbf{Q}_1 = \mathbf{Q}_2 = \mathbf{0}$. Case 1 stipulates only that the beam be on-axis at the final position. In case 2 we required that the beam state $\mathbf{x}_N$ be exactly $\mathbf{x}_f$. These objectives are enforced with

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P}_2 = \mathbf{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where $\mathbf{I}$ is the identity matrix. For both cases we see betatron oscillations throughout the channel. Case 1 exits on target, although with nonzero slope. In case 2 the beam leaves the channel on-axis with zero slope.

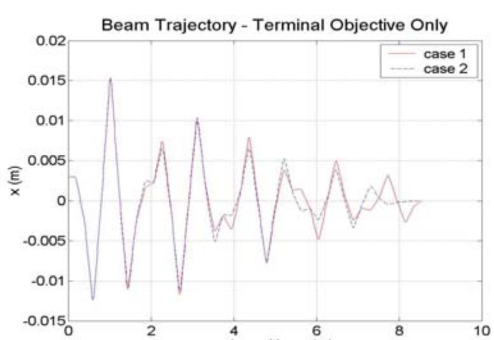


Figure 1: simulation cases 1 and 2

Case 3 and 4 are shown in Figure 2. In both these cases we enforce the final constraint $\mathbf{x}_N = \mathbf{x}_f$ by setting the terminal tuning matrix $\mathbf{P}_3 = \mathbf{P}_4 = \mathbf{I}$. Case 3 requires that the beam maintain proximity to the design axis throughout the channel while case 4 stipulates that the slope of the beam should minimized throughout. These two objectives are specified by choosing

$$\mathbf{Q}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

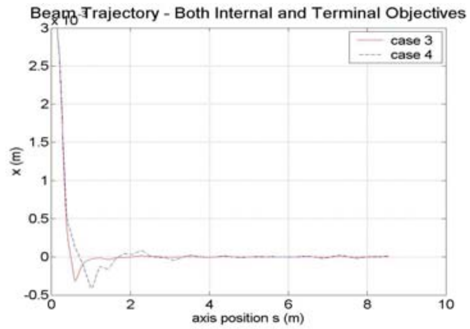


Figure 2: simulation cases 3 and 4

In Figure 2 we see that in case 3 the beam is brought onto the design axis quickly then maintains a close proximity to the axis. In case 4 the beam is also brought onto the axis, however not as rapidly. This result makes sense since we cannot act on the beam as strongly as in case 3.

## CONCLUSIONS AND FUTURE WORK

We have presented an alternative steering algorithm which, with the aid of a model, allows the user to specify beam behavior through a beamline, rather than just at specified BPM locations. It is also flexible enough to accommodate a variety of steering objectives simply by selecting different tuning parameters.

It would be interesting to apply a similar approach to the beam shaping problem. There the beam states would be the matrices of second-order moments, the controls would be the quadrupole strengths, and the cost functionals would describe shaping objectives for the beam. We speculate that the approach could be successful for a linear beam optics model.

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